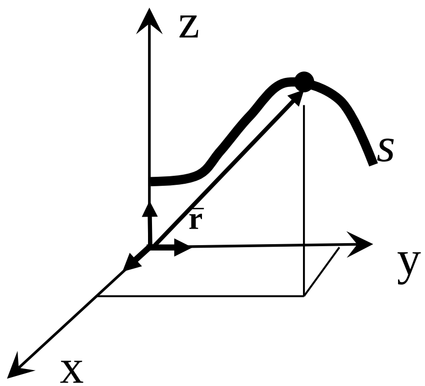
Exam Notes: **Particle Kinematics**

Particle Dynamics

Basic Kinematic Equations			Constant Acceleration
$v = \frac{ds}{dt}$	$\int_{t_o}^{t_1} v(t)dt = \int_{s_o}^{s_1} ds = s_1 - s_o$	$dt = \frac{ds}{v}$ $\quad t_1 - t_o = \int_{s_o}^{s_1} \frac{ds}{v(s)}$	$s_1 = s_0 + v_o(t_1 - t_o) + \frac{a}{2}\left(t_1^2 - t_o^2\right)$
$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$	$\int_{t_o}^{t_1} a(t)dt = \int_{v_o}^{v_1} dv = v_1 - v_o$	$dt = \frac{dv}{a}$ $\quad t_1 - t_o = \int_{v_o}^{v_1} \frac{dv}{a(v)}$	$v_1 = v_0 + a(t_1 - t_o)$
$ads = vdv$		$\int_{s_o}^{s_1} a(s)ds = \int_{v_o}^{v_1} vdv = \frac{1}{2}\left(v_1^2 - v_o^2\right)$	$v_1^2 = v_0^2 + 2a(s_1 - s_o)$

Rectangular Coordinates

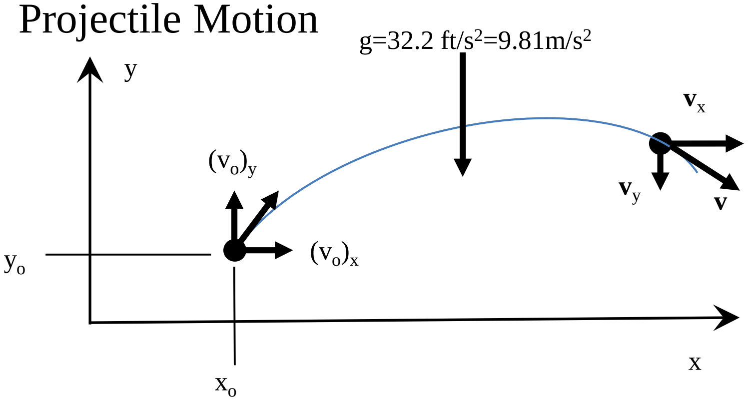


$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \qquad \mathbf{r} = r\mathbf{u}_r \qquad r = \sqrt{x^2 + y^2 + z^2}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}} \qquad v_x = \dot{x},\ \ v_y = \dot{y},\ \ v_z = \dot{z} \qquad v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}} \qquad a_x = \dot{v}_x,\ \ a_y = \dot{v}_y,\ \ a_z = \dot{v}_z \qquad a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

Projectile Motion



$$x = x_0 + (v_o)_x t$$

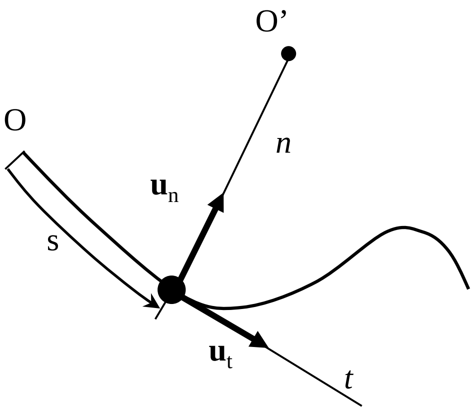
$$v_y = (v_o)_y - gt$$

$$y = y_o + (v_o)_y t - \frac{1}{2}gt^2$$

$$v_y^2 = v_o^2 - 2g(y - y_o)$$

Exam Notes: **Particle Kinematics**

Curvilinear Motion: Normal & Tangential Components



$$\mathbf{v} = v\mathbf{u}_t \qquad v = \dot{s}$$

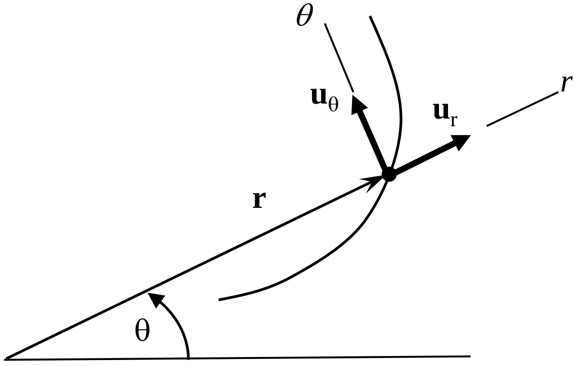
$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left|\frac{d^2y}{dx^2}\right|}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \dot{v}\mathbf{u}_t + \frac{v^2}{\rho}\mathbf{u}_n = a_t\mathbf{u}_t + a_n\mathbf{u}_n$$

$$a_t = \dot{v} \qquad a_n = \frac{v^2}{\rho} \qquad a = \sqrt{a_t^2 + a_n^2}$$

$$a_t ds = v dv$$

Curvilinear Motion: Cylindrical Components



$$\mathbf{r} = r\mathbf{u}_r$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta = v_r\mathbf{u}_r + v_\theta\mathbf{u}_\theta \qquad v = \sqrt{v_r^2 + v_\theta^2} = \sqrt{\dot{r}^2 + \left(r\dot{\theta}\right)^2}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = a_r\mathbf{u}_r + a_\theta\mathbf{u}_\theta \qquad a_r = \ddot{r} - r\dot{\theta}^2 \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \qquad a = \sqrt{a_r^2 + a_\theta^2}$$

Exam Notes: **Particle Kinematics**

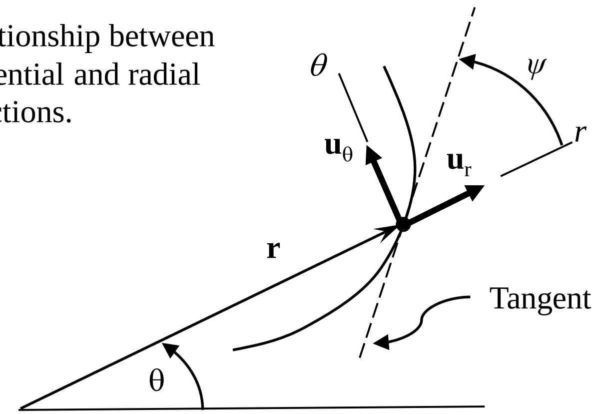
Relative-Motion; Translating Axes

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$$

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

Relationship between tangential and radial directions.



$$\tan\psi = \frac{r}{dr/d\theta}$$

ψ is counter-clockwise positive

Exam Notes: **Particle Kinetics**

Rectangular Coordinates

$$\sum F_x = ma_x$$

$$\sum F_y = ma_y$$

$$\sum F_z = ma_z$$

Normal & Tangential Coordinates

$$\sum F_t = ma_t$$

$$\sum F_n = ma_n$$

$$\sum F_z = 0$$

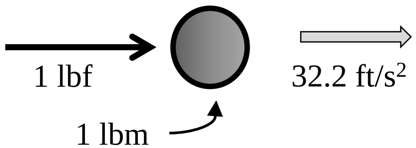
Cylindrical Coordinates

$$\sum F_r = ma_r$$

$$\sum F_\theta = ma_\theta$$

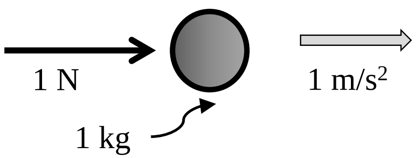
$$\sum F_z = ma_z$$

1 lb force accelerates a 1 lb mass 32.2 ft/s^2



$$1\,\text{lbf} = 32.2\,\frac{\text{lbm}\cdot\text{ft}}{\text{s}^2} \qquad 1\,\text{slug} = 32.2\,\text{lbm}$$

A 1 Newton force accelerates a 1 kg mass 1 m/s^2



$$1\,\text{N} = 1\,\frac{kg\cdot\text{m}}{\text{s}^2} \qquad 1\,\text{slug} = 32.2\,\text{lbm}$$

Acceleration due to gravity on earth

$$g = 32.2\,\frac{\text{ft}}{\text{s}^2} \qquad g = 9.81\,\frac{\text{m}}{\text{s}^2}$$

Exam Notes: **Particle Kinetics**

Work of a Force

$U_{1-2} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} = \int_{s_1}^{s_2} F \cos\theta ds$

Constant F along a straight line

$U_{1-2} = F_x \cos\theta (s_2 - s_1)$

Work of a Weight

$U_{1-2} = -W\Delta y$

Work of a Spring Force

$U_{1-2} = \int_{s_1}^{s_2} ksds = \frac{1}{2}ks_2^2 - \frac{1}{2}ks_1^2$

Work of Friction

$U_{1-2} = -\mu N \Delta s$

Principle of Work & Energy

$T_1 + \sum U_{1-2} = T_2$

where $T_1 = \frac{1}{2}mv_1^2$ $\quad U_{1-2} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r}$ $\quad T_2 = \frac{1}{2}mv_2^2$

For a system of particles $\quad \sum T_1 + \sum U_{1-2} = \sum T_2$

Power & Efficiency

$P = \frac{dU}{dt} = \mathbf{F} \cdot \mathbf{v}$

1 Watt= 1 J/s = 1 N·m/s

1 hp = 550 ft·lb/s

Efficiency $\quad \varepsilon = \frac{\text{power output}}{\text{power input}} = \frac{\text{energy output}}{\text{energy input}}$

Conservation of Energy

$T_1 + V_1 = T_2 + V_2$

where $T_1 = \frac{1}{2}mv_1^2$ $\quad T_2 = \frac{1}{2}mv_2^2$

For a system of particles $\quad \sum T_1 + \sum V_1 = \sum T_2 + \sum V_2$

V is the potential energies (elastic or gravitational)

Gravitational Potential

$V_g = mgh$

Elastic Potential

$V_e = \frac{1}{2}ks^2$

Linear Momentum

$m\mathbf{v}_1 + \sum \int \mathbf{F}dt = m\mathbf{v}_2$

System of Particles

$\sum m_i (\mathbf{v}_i)_1 + \sum \int \mathbf{F}_i dt = \sum m_i (\mathbf{v}_i)_2$

Conservation of Linear Momentum

$\sum m_i (\mathbf{v}_i)_1 = \sum m_i (\mathbf{v}_i)_2$

Impact

$m_A (v_A)_1 + m_B (v_B)_1 = m_A (v_A)_2 + m_B (v_B)_2$

Coefficient of Restitution

$e = \frac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$

Angular Momentum

$H_o = dmv$ $\quad$ Units: kg·m²/s or slug·ft²/s or lbf·ft·s

Vector Form

$\mathbf{H}_o = \mathbf{r} \times m\mathbf{v}$

$$\mathbf{H}_o = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ mv_x & mv_y & mv_z \end{vmatrix} = m(r_y v_z - r_z v_y)\hat{\mathbf{i}} - m(r_x v_z - r_z v_x)\hat{\mathbf{j}} + m(r_x v_y - r_y v_x)\hat{\mathbf{k}}$$

Relationship between Moment of a Force and the Rate of Change of Angular Momentum

$\sum \mathbf{M}_o = \dot{\mathbf{H}}_o$

Angular Impulse & Momentum

angular impulse $= \int \mathbf{M}_o dt = \sum (\mathbf{r} \times \mathbf{F}) dt$

$(\mathbf{H}_o)_1 + \sum \int \mathbf{M}_o dt = (\mathbf{H}_o)_2$

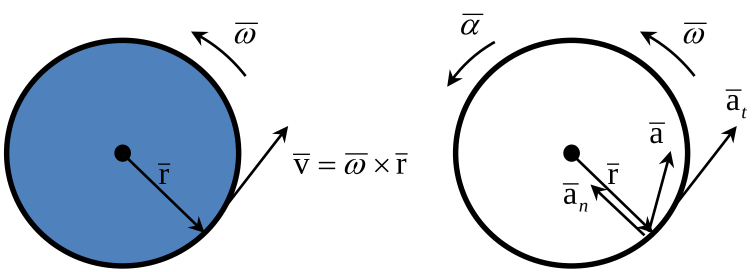
Conservation of Angular Momentum

$(\mathbf{H}_o)_1 = (\mathbf{H}_o)_2$

Exam Notes: **Planar Rigid Body Kinematics**

Basic Kinematic Equations				Constant Acceleration
$\omega = \frac{d\theta}{dt}$	$\int_{t_o}^{t_1} \omega(t)dt = \int_{\theta_o}^{\theta_1} d\theta = \theta_1 - \theta_o$	$dt = \frac{d\theta}{\omega}$ $\quad t_1 - t_o = \int_{\theta_o}^{\theta_1} \frac{d\theta}{\omega(\theta)}$		$\theta_1 = \theta_0 + \omega_o(t_1 - t_o) + \frac{\alpha_c}{2}\left(t_1^2 - t_o^2\right)$
$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$	$\int_{t_o}^{t_1} \alpha(t)dt = \int_{\omega_o}^{\omega_1} d\omega = \omega_1 - \omega_o$	$dt = \frac{d\omega}{\alpha}$ $\quad t_1 - t_o = \int_{\omega_o}^{\omega_1} \frac{d\omega}{\alpha(\omega)}$		$\omega_1 = \omega_0 + \alpha_c(t_1 - t_o)$
$\alpha d\theta = \omega d\omega$		$\int_{\theta_o}^{\theta_1} \alpha(\theta)d\theta = \int_{\omega_o}^{\omega_1} \omega d\omega = \frac{1}{2}\left(\omega_1^2 - \omega_o^2\right)$		$\omega_1^2 = \omega_0^2 + 2\alpha_c(\theta_1 - \theta_o)$

Rotation about a Fixed Point



$$\bar{a} = \bar{\alpha} \times \bar{r} - \omega^2 \bar{r}$$

$$\bar{a}_t = \bar{\alpha} \times \bar{r}$$

$$\bar{a}_n = -\omega^2 \bar{r}$$

Chain Rule for Rotational Motion

$$x = f(\theta)$$

$$\dot{x} = \frac{df}{dt} = \frac{df}{d\theta}\frac{d\theta}{dt} = \frac{df}{d\theta}\omega$$

$$\ddot{x} = \frac{d^2 f}{dt^2} = \frac{df}{d\theta}\frac{d^2\theta}{dt^2} + \frac{d^2 f}{d\theta^2}\left(\frac{d\theta}{dt}\right)^2 = \frac{df}{d\theta}\alpha + \frac{d^2 f}{d\theta^2}\omega^2$$

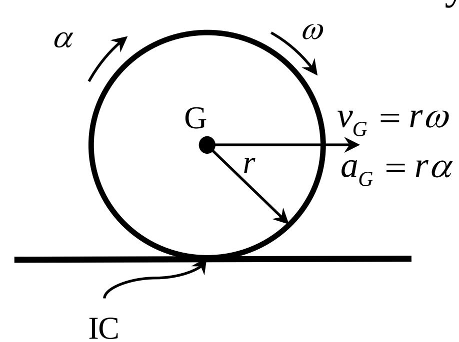
Relative Motion

$$\bar{r}_B = \bar{r}_A + \bar{r}_{B/A}$$

$$\bar{v}_B = \bar{v}_A + \bar{v}_{B/A} = \bar{v}_A + \bar{\omega} \times \bar{r}_{B/A}$$

$$\bar{a}_B = \bar{a}_A + \left(\bar{a}_{B/A}\right)_t + \left(\bar{a}_{B/A}\right)_n = \bar{a}_A + \bar{\alpha} \times \bar{r}_{B/A} - \omega^2 \bar{r}_{B/A}$$

Instantaneous Center of Zero Velocity



Exam Notes: **Planar Rigid Body Kinetics**

Mass Moment of Inertia

$$I = \int_m r^2 dm = \int_V r^2 \rho dV$$

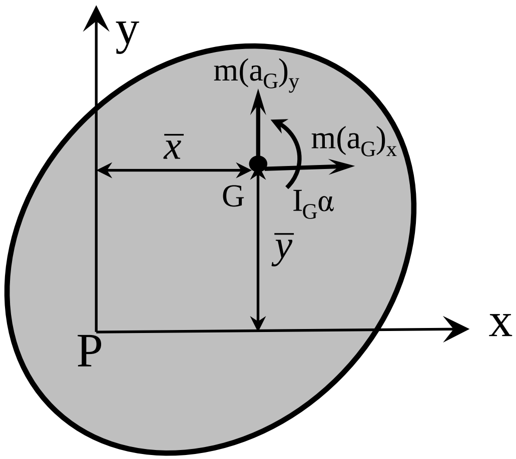
Radius of Gyration

$$I = mk^2$$

Parallel-Axis Theorem

$$I = I_G + md^2$$

General Plane Motion



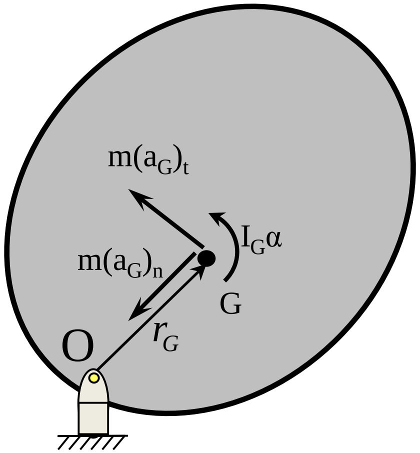
$$\sum F_x = m(a_G)_x$$

$$\sum F_y = m(a_G)_y$$

$$\sum M_G = I_G\alpha \qquad \sum M_{IC} = I_{IC}\alpha$$

$$\sum M_P = \sum (\mathcal{M}_k)_P = -\bar{y}m(a_G)_x + \bar{x}m(a_G)_y + I_G\alpha$$

Rotation about a Fixed Point



$$\sum F_n = m(a_G)_n = m\omega^2 r_G$$

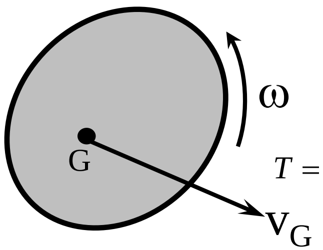
$$\sum F_t = m(a_G)_t = m\alpha r_G$$

$$\sum M_G = I_G\alpha \qquad \sum M_o = I_o\alpha$$

Exam Notes: **Planar Rigid Body Kinetics: Work & Energy**

Kinetic Energy

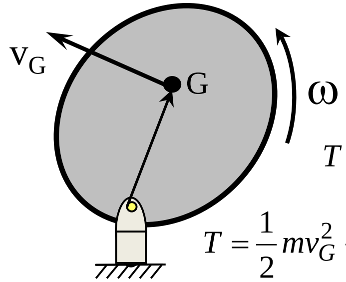
General Plane Motion



$$T = \frac{1}{2} I_{IC}\omega^2$$

$$T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$$

Rotation about a Fixed Axis



$$T = \frac{1}{2} I_o \omega^2$$

$$T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$$

Work

$$U_M = \int_{\theta_1}^{\theta_2} M d\theta \qquad U_W = -W\Delta y \qquad U_s = \frac{1}{2} k_s \left(s_2^2 - s_1^2\right)$$

Principle of Work & Energy

$$T_1 + \sum U_{1-2} = T_2$$

Potential Energy

Gravitational Potential $V_g = mgh$

Elastic Potential $V_e = \frac{1}{2} k s^2$

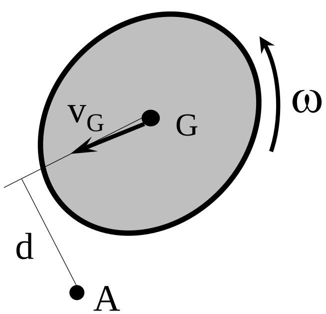
Conservation of Energy

$$T_1 + V_1 = T_2 + V_2$$

Exam Notes: **Planar Rigid Body Kinetics: Impulse & Momentum**

Angular Momentum

General Plane Motion

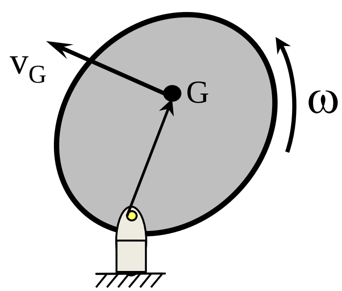


$$H_A = I_G\omega + d m v_G$$

$$H_G = I_G\omega \qquad H_{IC} = I_{IC}\omega$$

$$L = m v_G$$

Rotation about a Fixed Axis



$$H_G = I_G\omega \qquad H_o = I_o\omega$$

$$L = m v_G$$

Linear Impulse & Momentum

$$m(\mathbf{v_G})_1 + \sum \int \mathbf{F} dt = m(\mathbf{v_G})_2$$

$$I_G\omega_1 + \sum \int \mathbf{M}_G dt = I_G\omega_2$$

$$I_o\omega_1 + \sum \int \mathbf{M}_o dt = I_o\omega_2$$

Conservation of Momentum

$$\sum m_i (\mathbf{v}_{Gi})_\mathbf{1} = \sum m_i (\mathbf{v}_{Gi})_\mathbf{2}$$

$$\sum (H_A)_1 = \sum (H_A)_2$$

$$\sum (H_G)_1 = \sum (H_G)_2$$