Quadcopter Simulation

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Abstract - The equations of motion for a quadcopter are derived, starting with the voltage torque relation for the brushless motors and then going through the quadcopter kinematics and dynamics. Aerodynamic effects like blade-flapping and non-zero free streem velocity are ignored but air friction as a linear drag force in all directions is included. A simulation is created using the equations of motion in order to test and visualize quadcopter control mechanisms. A PID controller (with minor modifications to prevent wind-up) is applied on the quadcopter to make it robust against disturbances.

I. INTRODUCTION

A quadrotor helicopter (quadcopter) is a helicopter that has four rotors that are spaced equally from each other. They are usually at the corners of a square body. Quadcopters have become really popular recently because of the increase in technology. In the past it was very difficult to control four independent rotors without electronic assistance. Because microprocessors have become far less expensive, electronic and autonomous control of quadcopters has become possible for commercial, military, and personal uses.

Controlling quadcopters is a difficult but interesting problem. They have six degrees of freedom, three translational and three rotational, but only four independent inputs (rotor speeds), thus creating an underactuated system. Translational and rotational motion is couples in order to achieve six degrees of freedom. This results in highly nonlinear dynamics, especially when taking into account complicated aerodynamic effects.

II. QUADCOPTER DYNAMICS

There are two coordinate frames that the quadcopter will operate in. The inertial frame is defined by the ground with the z direction pointing opposite of gravity. The body frame is attached to the orientation of the quadcopter with the rotor axes pointing in the positive z direction and the arms pointing in the x and y directions.

A. Kinematics

The position and velocity of the quadcopter are defined in the inertial frame as $x = (x,y,z)^T$ and $\dot{x} = (\dot{x}, \dot{y}, \dot{z})^T$. The roll, pitch and yaw angles are in the body frame as $\theta = (\phi, \theta, \psi)^T$, and the angular velocities are equal to $\dot{\theta} = \left(\dot{\phi}, \dot{\theta}, \dot{\psi}\right)^T$. The angular velocity vector $\omega \neq \dot{\theta}$. The angular velocity is a vector pointing along the axis of rotation, while $\dot{\theta}$ is just the time derivative of yaw, pitch, and roll. The rotation matrix in (1) is needed to convert these angular velocities into the angular velocity vector.

$$\omega = \begin{bmatrix} 1 & 0 & -S\theta \\ 0 & C\phi & C\theta S\phi \\ 0 & -S\phi & C\theta C\phi \end{bmatrix} \dot{\theta} \qquad (1)$$

Where ω is the angular velocity in the body frame. In order to go from the body frame to the inertial frame the rotation matrix in (2) is used. This matrix is derived using the ZYZ Euler angle conventions.

$$R = \begin{bmatrix} C\phi C\psi - C\theta S\phi S\psi & -C\phi S\phi - C\phi C\theta\, S\psi & S\theta S\phi \\ C\theta C\psi S\phi & C\phi C\theta C\psi - S\phi S\psi & -C\psi S\theta \\ S\phi S\theta & C\phi S\theta & C\theta \end{bmatrix} \qquad (2)$$

B. Physics

An understanding of the physical properties that govern the system are needed in order to properly model the dynamics of the system. First a description of the motors being used will be given, and then energy will be taken into account to derive the forces and thrusts that the motors produce on the entire quadcopter. Since all the motors on the quadcopter are identical only on will need to be analyzed. The propellers are configured in a way that the adjacent propellers spin in opposite directions of each other. This balances the torques if all the propellers are spinning at the same rate.

C. Motors

All quadcopters use brushless motors. The torque produced in an electric motor is given by (3).

$$\tau = k_t(I - I_0) \tag{3}$$

Where τ is the motor torque, I is the current input, I_0 is the current with no load on the motor, and k_t is the torque proportionality constant. The voltage that is measured across the motor is equal to the sum of the back-EMF and resistive loss as shown in (4).

$$V = IR_m + K_v\omega \tag{4}$$

Where V is the voltage drop across the motor, R_m is the motor resistance, ω is the angular velocity of the motor, and k_v is a proportionality constant (indicating back-EMF generated per RPM). Using (3) and (4) the power the motor consumes can be calculated as shown in (5).

$$P = IV = \frac{(\tau + K_t I_0)(K_t I_0 R_m + \tau R_m + K_t K_v \omega)}{K_t^2} \tag{5}$$

To simplify the model, the motor resistance in (6) is neglected.

$$P = \frac{(\tau + k_t I_0) K_v \omega}{K_t} \tag{6}$$

In simplifying the model even further, the term $k_t I_0$ is assumed to be much smaller than τ which is reasonable since I_0 is very small. So the final simplified equation for power is shown in (7).

$$P = \frac{K_v}{K_t}\tau\omega \tag{7}$$

D. Forces

From conservation of energy, the energy the motor expends in a given time period is equal to the force generated on the propeller times the distance that the air it displaces moves. equivalently, the power is equal to the thrust times air velocity as shown in (8).

$$P = TV_h \tag{8}$$

Assuming vehicle speeds are low, so V_h is the air velocity when the quadcopter is hovering. The free stream velocity is also assumed to be zero. Momentum theory gives the equation for hover velocity as a function of thrust.

$$V_h = \sqrt{\left(\frac{T}{2\rho A}\right)} \tag{9}$$

in (9) ρ is the density of the surrounding air and A is the area swept out by the rotor. Combining (7) and (9) results in (10).

$$P = \frac{T^{\frac{3}{2}}}{\sqrt{2\rho A}} \tag{10}$$

Where the torque is proportional to the thrust T by a constant ratio k_τ determined by the blade configuration and parameters. To solve for the thrust magnitude T, it is found that thrust is proportional to the square of angular velocity of the motor as shown in (11).

$$T = \left(\frac{k_v K_\tau \sqrt{2\rho A}}{k_t}\omega\right)^2 = k\omega^2 \tag{11}$$

If all the motors are summed, the total thrust on the quadcopter in the body frame is given by (12).

$$T_B = k\begin{bmatrix} 0 \\ 0 \\ \sum \omega_i^2 \end{bmatrix} \tag{12}$$

Along with the thrust force, friction as a force proportional to the linear velocity is modeled. (13) is a simplified model of fluid friction, but will be good enough for the model.

$$F_D = [-k_d\dot{x}, -k_d\dot{y}, -k_d\dot{z}]^T \tag{13}$$

E. Torques

All the forces acting on the quadcopter have been computed, now the torques will also be computed. Each rotor adds torque about the z body z axis. This torque is the torque required to keep the propeller spinning while providing thrust. The drag equation form fluid dynamics is used in (14).

$$F_d = \frac{1}{2}\rho C_D A v^2 \tag{14}$$

Where ρ is the surrounding fluid density, A is the propeller cross section and C_D is a dimensionless constant. Then the torque due to the drag is given by (15) .

$$\tau_D = \frac{1}{2} R\rho C_D A v^2 = \frac{1}{2} R\rho C_D A(\omega R)^2 = b\omega^2 \quad (15)$$

Where ω is the angular velocity of the propeller, R is the radius of the propeller, and b is a constant. The total torque about the z axis is given in (16) by summing of all the torques from each propeller.

$$\tau_\psi = b(\omega_1^2 - \omega_2^2 + \omega_3^2 - \omega_4^2) \quad (16)$$

The roll and pitch torques are taken from standard mechanics. The pitch torque is given by (17).

$$\tau_\phi = LK(\omega_1^2 - \omega_3^2) \quad (17)$$

Where L is the distance from the center of the quadcopter to any of the propellers. Adding all of these together gives (18), total torques in the body frame.

$$\tau_B = \begin{bmatrix} LK(\omega_1^2 - \omega_3^2) \\ LK(\omega_2^2 - \omega_4^2) \\ b(\omega_1^2 - \omega_2^2 + \omega_3^2 - \omega_4^2) \end{bmatrix} \quad (18)$$

This model that has been derived is a highly simplified model. Many advanced effects are ignored that make the model highly nonlinear. Rotational drag forces, blade flapping, wind, etc. are ignored.

F. Equations of Motion

The acceleration of the quadcopter in the inertial frame is due to thrust, gravity, and linear friction. So the linear motion can be expressed by (19).

$$m\ddot{x} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + RT_B + F_D \quad (19)$$

Where x is the position of the quadcopter, g is the acceleration from gravity, F_D is the drag force, and T_B is the thrust vector in the body frame.

It is helpful to have the linear equations of motion in the inertial frame, but the rotational equations of motion are more easily written in the body frame, so that rotations can be expressed about the center of the quadcopter. The rotational equations of motion are found in (20) which are derived from Euler's equations for rigid body dynamics.

$$\dot{\omega} = \begin{bmatrix} \dot{\omega_x} \\ \dot{\omega_y} \\ \dot{\omega_z} \end{bmatrix} = I^{-1}\big(\tau - \omega \times (I\omega)\big) \quad (20)$$

Where ω is the angular velocity vector, I is the moment of inertia, and τ is a vector of external torques. To obtain I, the quadcopter is modeled as two thin uniform rods crossed at the origin with a point mass (motor) at the end of each rod. This produces (21) a diagonal inertia matrix.

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \quad (21)$$

The final equation for the rotations in the body frame are found in (22).

$$\dot{\omega} = \begin{bmatrix} \tau_\phi I_{xx}^{-1} \\ \tau_\theta I_{yy}^{-1} \\ \tau_\psi I_{zz}^{-1} \end{bmatrix} - \begin{bmatrix} \frac{I_{yy} - I_{zz}}{I_{xx}} \omega_y \omega_z \\ \frac{(I_{zz} - I_{xx})}{I_{yy}} \omega_x \omega_z \\ \frac{I_{xx} - I_{yy}}{I_{zz}} \omega_x \omega_y \end{bmatrix} \quad (22)$$

III. CONTROL

The purpose of getting a mathematical model of a quadcopter is to help develop controllers for real quadcopters. The inputs of the system are the four angular velocities of the rotors. The system can be written as a first order differential equation in state space form as shown in (23).

$$\begin{aligned}
\dot{x_1} &= x_2 \\
\dot{x_2} &= \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} + \frac{1}{m} RT_B + \frac{1}{m} F_D \\
\dot{x_3} &= \begin{bmatrix} 1 & 0 & -S\theta \\ 0 & C\phi & C\theta S\phi \\ 0 & -S\phi & C\theta C\phi \end{bmatrix}^{-1} x_4 \\
\dot{x_4} &= \begin{bmatrix} \tau_\phi I_{xx}^{-1} \\ \tau_\theta I_{yy}^{-1} \\ \tau_\psi I_{zz}^{-1} \end{bmatrix} - \begin{bmatrix} \frac{I_{yy} - I_{zz}}{I_{xx}} \omega_y \omega_z \\ \frac{(I_{zz} - I_{xx})}{I_{yy}} \omega_x \omega_z \\ \frac{I_{xx} - I_{yy}}{I_{zz}} \omega_x \omega_y \end{bmatrix}
\end{aligned} \quad (23)$$

where x_1 is the quadcopter position, x_2 is the linear velocity, x_3 is the roll, pitch and yaw angles, and x_4 is the angular velocity vector. All of these are vectors with three elements.

In order to control the quadcopter, a PID controller will be used, with a component proportional to the error between the desired trajectory and the observed trajectory, and a component proportional to the derivative of the error. The quadcopter will be stabilized in a horizontal position, so the desired velocities and angles will all be zero. The torques will be set proportional to the output of the controller as shown in (24).

$$\begin{bmatrix}\tau_\phi\\ \tau_\theta\\ \tau_\psi\end{bmatrix} = \begin{bmatrix}-I_{xx}(k_d\dot{\phi} + k_p\int_0^T \dot{\phi}dt\\ -I_{yy}(k_d\dot{\theta} + k_p\int_0^T \dot{\theta}dt\\ -I_{zz}(k_d\dot{\psi} + k_p\int_0^T \dot{\psi}dt\end{bmatrix} \tag{24}$$

Previously the relationship between torque and the inputs were derived which leads to (25).

$$\tau_B = \begin{bmatrix}LK(\omega_1^2 - \omega_3^2)\\ LK(\omega_2^2 - \omega_4^2)\\ b(\omega_1^2 - \omega_2^2 + \omega_3^2 - \omega_4^2)\end{bmatrix} = \begin{bmatrix}-I_{xx}(k_d\dot{\phi} + k_p\int_0^T \dot{\phi}dt\\ -I_{yy}(k_d\dot{\theta} + k_p\int_0^T \dot{\theta}dt\\ -I_{zz}(k_d\dot{\psi} + k_p\int_0^T \dot{\psi}dt\end{bmatrix} \tag{25}$$

This gives three equations with four unknowns. The fourth equation is found by setting the inputs to keep the quadcopter hovering, so that T = mg. The thrust points along the positive z axis which leads to (26).

$$T = \frac{mg}{cos\theta cos\phi} \tag{26}$$

With this extra constraint, four linear equations with four unknowns γ_i are obtained as shown in (27).

$$\begin{aligned}\gamma_1 &= \frac{mg}{4kcos\theta cos\phi} - \frac{2be_\phi I_{xx} + e_\psi I_{zz}kL}{4bkL}\\ \gamma_2 &= \frac{mg}{4kcos\theta cos\phi} + \frac{e_\psi I_{zz}}{4b} - \frac{e_\theta I_{yy}}{2kL}\\ \gamma_3 &= \frac{mg}{4kcos\theta cos\phi} - \frac{-2be_\phi I_{xx} + e_\psi I_{zz}kL}{4bkL}\\ \gamma_4 &= \frac{mg}{4kcos\theta cos\phi} + \frac{e_\psi I_{zz}}{4b} + \frac{e_\theta I_{yy}}{2kL}\end{aligned} \tag{27}$$

Where γ_i is the angular velocity of each motor squared, and e is the error. The errors are calculated in (28).

$$\begin{aligned}e_\phi &= k_d\dot{\phi} + k_p\int_0^T \dot{\phi}dt + k_i\int_0^T\int_0^T \dot{\phi}dtdt\\ e_\theta &= k_d\dot{\theta} + k_p\int_0^T \dot{\theta}dt + k_i\int_0^T\int_0^T \dot{\theta}dtdt\\ e_\psi &= k_d\dot{\psi} + k_p\int_0^T \dot{\psi}dt + k_i\int_0^T\int_0^T \dot{\psi}dtdt\end{aligned} \tag{28}$$

IV. SIMULATION

With the complete equations of motion and the controller in place, a simulation was built using simulink as shown in Fig. 1 and Fig. 2.

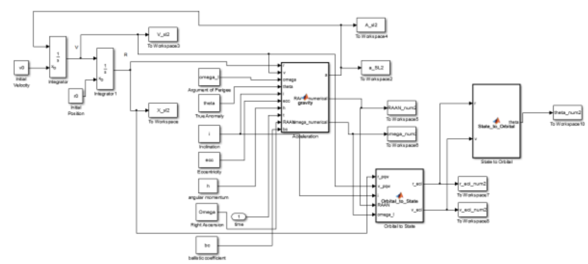

Fig. 1. Simulink Model

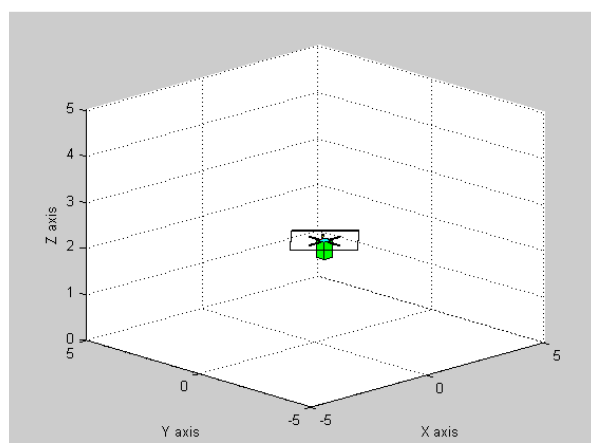

Fig. 2. Quadcopter Simulation