SURFACE WAVE PROPAGATION IN A DIELECTRIC WAVEGUIDE LOADED WITH AN ANISOTROPIC, CONDUCTIVE, AND SPATIALLY DISPERSIVE SUBSTRATE

by

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Abstract

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This thesis presents an analytical treatment of surface waves inside a dielectric slab loaded with a conductive and spatially dispersive semiconductor-like substrate. The work is primarily focused on the modelling of the substrate and getting the field solutions out from the Helmholtz equation. Appropriate boundary conditions have been used in order to get a unique dispersion relation. The surface wave modes are then extracted from the relation by using a root-searching algorithm, which in this work is the MATLAB Genetic Algorithm toolbox. Many different substrate configurations have been considered, starting from the very basic isotropic case to the most complex spatial dispersion case. In between, anisotropicity has also been added to the substrate by turning the static magnetic field on. The permittivity tensors are derived from the fluid transport equations and through the course of the thesis, extra terms such as plasma oscillations, damping, cyclotron resonance, and density perturbations are added. Many assumptions, approximations, and limitations of this analytical treatment have also been addressed. Simulations results have been shown to see the effects of these various terms. The substrates analyzed in the chapters are only a theoritical approximation of an actual substrate. The main idea behind this study is to get a feel for how the transport equations can be utilized to obtain properties that might be on

a macroscopic scale. The physical significance of this expose has also been discussed in the last chapter. Issues such as scalability to space plasmas and future ramifications are also included. The study done thus far will be useful in investigating such plasma mediums.

(159 pages)

To my beloved parents, sister, parrot, my grandparents, and all my friends in Workshop................

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Tushar Andriyas

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Chapter 1

Introduction

Surface wave analysis has always been a subject of great interest and diverse opinions, ranging from being present only in a theoretical sense and then being actually detected in practical situatuons [1, 2]. They exist in different domains of investigation and dimensions. Examples include ocean surface waves [3], ultrasonics [4], surface waves in space plasmas [5–7], down to being present in the micro and sub-micro level chips and antennas [8–10]. In electromagnetics, these waves were first analyzed by J. Zenneck in 1907 [11]. A surface wave exists at a sharp discontinuity between two different mediums, that can be either isotropic or anisotropic, permeable or nonpermeable, and gyroelectric or gyromagnetic media [12–16]. Many different modes can exist at the interface of different media, surface wave mode being one of them. In the past, these modes were thought to exist only theoretically [1], but they have been detected in many practical situations, such as solar wind driven waves that cause generation of surface currents in the bow-shock region of the magnetosphere [5], surface wave loss in a patch antenna [9, 17, 18], and even in seismic tremors. These waves have a peculiar property analogous to skin depth in metals [2,9,10,19]. They are confined to a very small region along the discontinuity and decay exponentially in direction perpendicular to the discontinuity in both the mediums and travel parallel to the interface [1, 2].

There are many techniques through which one can analyze these waves. Full wave simulation tools such as Method of Moments (MoM), Finite Element Method (FEM), and Finite-Difference Time-Domain (FDTD) [20] are used to draw an overall picture as to how the fields are effected at an interface in the presence of such waves [9, 21, 22]. But such tools are of no use until we have a theoretical background of what to look for. To an extent, analytical techniques such as dyadic Green's functions [9, 19, 23–27] can be approximated to simpler forms, but even these simpler forms can be complex, depending how the material

is described macroscopically. Another way to analyze these waves is by postulating fields on one side of the discontinuity and using Snel's law of reflection and refraction [21], the reflected and refracted fields in the two media can be calculated by using specific forms of field solutions [21, 28–31]. This technique can be used to analyze media which have isotropic behavior through TE/TM decomposition, as shown in fig. 1.1 [21].

Dispersion relation is one of the methods used to gain some insight into what conditions lead to a surface wave [2, 12, 32]. These analytical techniques require assumptions on what kind of field solutions can be typified as surface waves. Knowing the field solutions and using the wave equations in the two mediums, the dispersion relation can be found after matching of fields at the interface [8, 33]. The time dependence is always assumed to be sinusoidal $(e^{-j\omega t})$ and the spatial dependence is assumed as a traveling wave solution $(e^{j(k_y y + k_z z)})$ in the directions parallel to the interface and exponentially decaying perpendicular to the interface $(e^{-\zeta x}$, where ζ is the decay constant in either medium). This form is generally called a guided wave solution [14, 16, 19].

Mediums, at the interface of which these waves exist, can be finite, infinite, or semi-infinite in one medium [14, 28, 34–37]. The form of field solutions are different for such mediums. They can have either isotropic [2] or anisotropic behavior [32, 38], both uniaxial and biaxial [39]. Most of the work done till date has been focused on analyzing materials that are nonmagnetic, but some papers have taken the magnetic behavior into account, also [13, 16].

The analysis done can be put to practical use in the field of antenna theory [9, 17, 40, 41], understanding how a laboratory two plasma system can behave under different assumptions such as finite damping, presence of a steady magnetic field and finite pressure (or presence of diffusion) [42]. Also, it can be used to determine how different signatures of surface waves in magnetosphere suggest different structural properties of the bow shock, magnetosheath, and magnetopause interfaces [5].

The main focus of this work is to analyze how different material properties of the substrate of a patch antenna leads to different surface modes being supported at the interface.
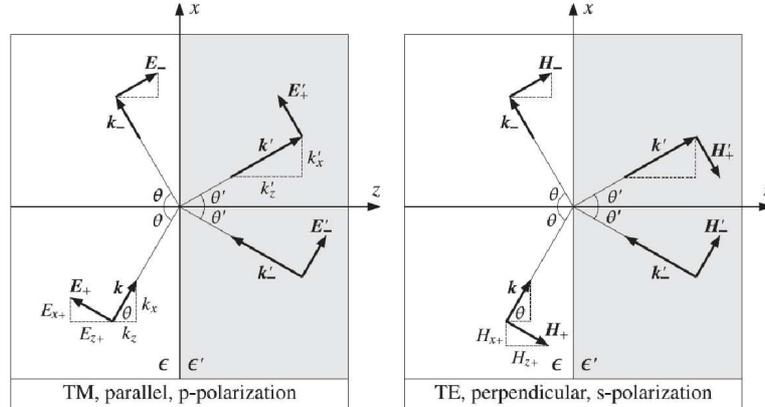
Fig. 1.1: Analysis of surface waves using reflection-refraction technique.

For patch antennas, these surface wave modes have a big part to play in reducing the radiation efficiency of the antenna [9, 17, 27, 43]. These modes can be observed as back lobes in the radiation pattern of a patch antenna [10], as depicted in fig. 1.2. While doing full wave analysis through Green's function, the presence of such modes introduces poles in the integrand [9, 37], shown in fig. 1.3, which in turn brings down the performance of the antenna as a radiator.

Many studies have therefore been done to reduce such interference of surface waves. Surfaces such as high impedance surfaces [44], have shown that these modes can be mitigated to an extent. There are asymptotic techniques elsewhere in the literature through which reduced surface wave patches can be built which increase the leaky wave coupling and enhance the radiation efficiency [45]. Most of the investigations with regards to studying surface waves in anisotropic mediums have already been done, but many assumptions have been taken, whether it be neglecting the damping term [46] or density perturbation [12, 29]. When density perturbations were included in the material parameters, the substrate permittivity ($\tilde{\epsilon}_p$) was assumed to be a scalar rather than a tensor [47], while in many other papers, the substrate material with isotropic behavior and layered inhomogeneous material
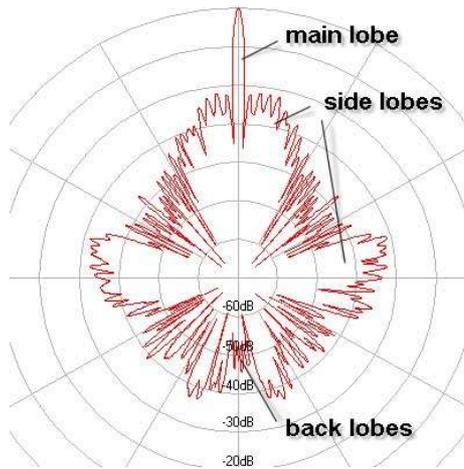
Fig. 1.2: Radiation pattern of a practical antenna.

with different inhomogeneity profiles was studied [43, 48]. This thesis will therefore aim at finding dispersion equations for such cases and verify the results against the specific cases already in literature. Also, emphasis will be on investigating the surface wave behavior with different material properties through dispersion curves rather than studying the effect of these modes on the overall performance of a patch antenna.

1.1 Definition

Surface waves are characterized by a field that decays exponentially away from an interface between two different media, most of which is contained in or near the dielectric [2]. By different media, we mean the media should have different material parameters. Macroscopically speaking, they should have a different ϵ and/or μ. Throughout the work, both the materials will be assumed nonmagnetic and it is only the permittivity that will be worked upon. As one will find out during the course of this chapter, there are certain conditions that should be met in order for a wave solution to be called a surface wave. From an antenna prospective, these waves account for the loss along with metallic and dielectric loss for an antenna [9, 10, 18, 40], unless the antenna is a reduced surface wave antenna [45]. In other words, these waves are responsible for the lowering of the radiated antenna power.

In this work, we are not really interested in finding out how much power is actually lost through the surface wave mechanism or how to minimize such waves (which has been done
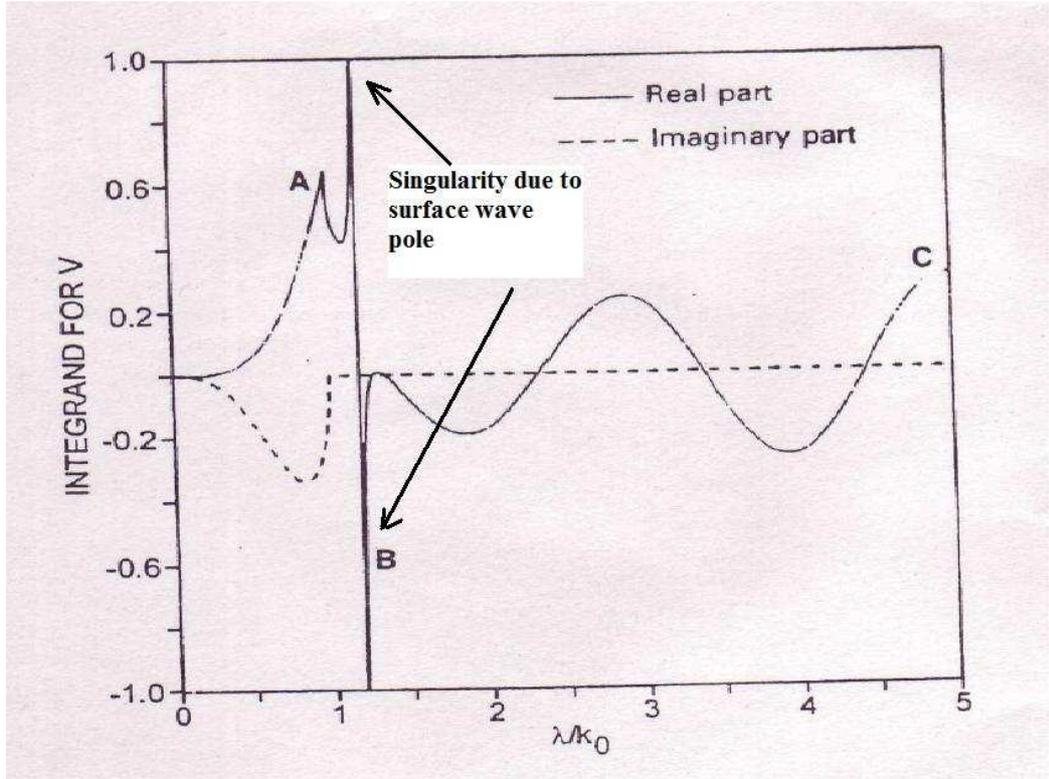
Fig. 1.3: Real and imaginary part of the integrand of the Green's function.

elsewhere [9, 10]), but to find these wave modes in different configurations of semiconductor media, viz. isotropic frequency dependent, anisotropic frequency dependent, isotropic frequency and wave vector $(\vec{k})$ dependent, and anisotropic frequency and wave vector dependent. The anisotropic case will include different configurations of steady magnetic field $\vec{B}_0$ and $\vec{k}$ such as Voigt and Faraday geometries [49, 50].

1.2 Dispersion Relation

The method used to find the surface modes is through the derivation of dispersion relation [2], which basically is a relation between the traveling wave vector $\vec{k}$ and the input frequency ω. In vacuum, this is given by

$$k = \frac{\omega}{c},\tag{1.1}$$

where c is the speed of light [8, 24, 33]. Equation (1.1) means that k and ω have a linear relationship, i.e., vacuum is dispersionless meaning that all frequencies will travel with the same phase velocities and there would be no distortion of signal shape [2]. But when the material parameters are frequency dependent, $\vec{k}$ is no longer linearly related to the input frequency [51], implying that different frequencies would travel at different phase velocities. Therefore, by studying the dispersion relation, we can judge how different frequencies would propagate in a given medium and at an interface between different media.

When there are two different media, each having different properties, the dispersion relation is found by solving the homogeneous wave equation in the two media and matching them across the interface with appropriate boundary conditions. It is of prime importance to employ the right boundary conditions or the solution would not be unique and complete [52]. Since dispersion relation is an algebraic expression, the fields can be solved for as a function of temporal and spatial frequency. Before starting the derivation, one therefore has to assume some form of a solution for the field quantities, in order to make the space derivatives into spatial frequencies [12, 19, 22, 28, 53]. For example, a form such as

$$\vec{A}(x, y, z, t) = \vec{A_0} e^{-\alpha x} e^{j(k_y y + k_z z - \omega t)} \tag{1.2}$$

for medium 1, and

$$\vec{A}(x, y, z, t) = \vec{A_1} e^{-\beta x} e^{j(k_y y + k_z z - \omega t)} \tag{1.3}$$

for medium 2, would transform the time and space derivatives as $\frac{\partial}{\partial t} \Rightarrow -j\omega$, $\frac{\partial}{\partial x} \Rightarrow -\alpha$, $\frac{\partial}{\partial y} \Rightarrow jk_y$, and $\frac{\partial}{\partial z} \Rightarrow jk_z$. This will transform the differential form of Maxwell's curl equations, i.e.,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{1.4}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}, \tag{1.5}$$

into algebraic equations, as per the conventions given (where $\vec{A}$ is field quantity (either $\vec{E}$ or

$\vec{H}$)) and should not be confused with the spatial Fourier transform done in Chapter 5 [12,32]. Looking at the form given in eqs. (1.2) and (1.3), given that x is the spatial coordinate perpendicular to the interface, α would have to be a positive real quantity or a complex quantity with positive real part, for the solution to be a surface wave. α and β are the respective solutions of the wave equation in medium 1 and 2. The number of solutions, α_i and β_j, will depend on power of the characteristic equation. In other words, they will depend on the order of the derivative in x, i.e., $\frac{\partial}{\partial x}$. For an isotropic medium the wave equation is of second order and so i or j would be 2 [2]. But when the medium is anisotropic, the wave equation is of the fourth order implying that the characteristic equation would have four solutions [12, 29]. More details about the roots and their properties would be easy to understand in the forthcoming chapters.

1.3   Patch Antenna

A patch antenna is a low-profile resonant structure which has many advantages in terms of size, weight, and ease of integration with other electronics [10]. The basic construction of a patch is shown in the fig. 1.4. A substrate is sandwiched between the ground plane and the radiating element. The properties of this substrate determine the resonant frequency of the patch. The radiation is in the broadside direction, i.e., in x direction.

Reduction in the radiation efficiency of a patch is mainly due substrate losses, copper losses (due to the ground plane and the radiator), due to surface wave loss, and can also be affected by spurious radiation from the feed itself [9]. These losses can be seen as minor lobes in the radiation pattern (fig. 1.2). A surface wave decays as inverse square root of distance from the excitation as opposed to space waves which follow inverse square law [9,11]. Surface waves are excited in a waveguide structure (such as a patch antenna), when $\epsilon_r > 1$. These are launched at angle $\frac{\pi}{2} < \phi < sin^{-1}(\frac{1}{\sqrt{\epsilon_r}})$. When there lies a discontinuity in the propagation direction (which is the case with most finite structures), as shown in fig. 1.5, they get partly reflected and diffracted, the latter being responsible for end-fire radiation (or reduction in radiation efficiency of a patch) [9]. The main focus of this work is analyzing surface wave modes in a dielectric slab loaded with a complex substrate such as that of
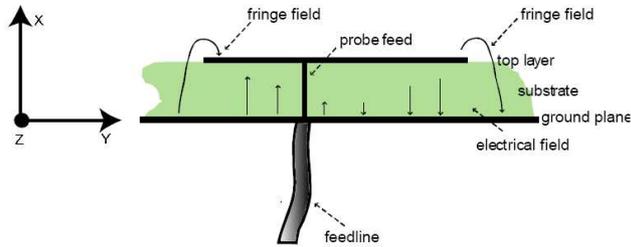
Fig. 1.4: A coaxially fed patch antenna.

Pozar [2].

1.4 Isotropic Frequency Independent Media and Issues with Field Solutions

The generic example that helps in gaining some insight on how a dispersion relation works is of a waveguide filled with isotropic media of relative permittivity ϵ_r and surrounded by free space [2]. The waveguide is taken to be infinite in the interface plane, i.e., along y and z directions. Solving the wave equation in both the media, we get the attenuation constants in the dielectric and free space as

$$\alpha^2 = \epsilon_r k_0^2 - k_z^2, \tag{1.6}$$

$$\beta^2 = k_z^2 - k_0^2, \tag{1.7}$$

respectively, where the wave numbers and attenuation constants have been defined before. k_0 is the free space wave equal to $\frac{\omega}{c}$. As mentioned in Pozar [2], α and β have been chosen such that the x dependence of the field solutions inside the waveguide and in free space are of the form $A sin(\alpha x) + B cos(\alpha x)$ and $Ce^{\beta x} + De^{-\beta x}$, respectively. The solution put forth by Pozar [2] is thus one of the many solutions that can exist in the waveguide. Also, the solution inside the waveguide is based on the fact that the backward and forward
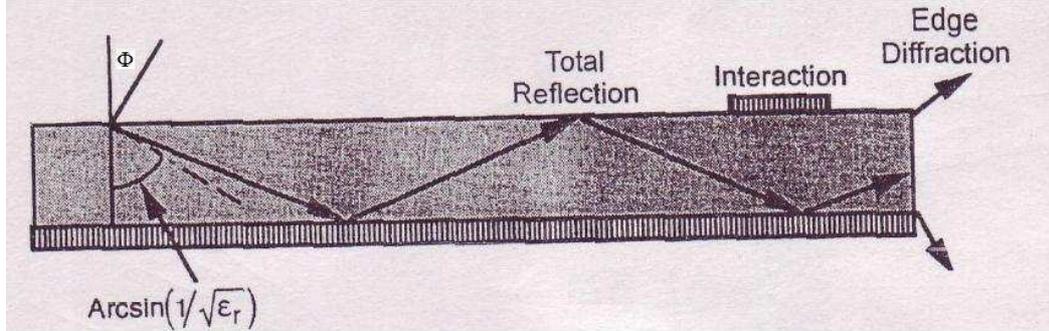
Fig. 1.5: Excitation of surface waves in patch antennas.

traveling waves have the same amplitude, which may not be the case in general. Although the dispersion relation should be for the most general field forms, at times, because of the complexity involved with the derivations of the wave equation inside the substrate, the final relation would be a big square matrix, the determinant of which would be hard to find, even computationally [12, 31, 47]. During the next chapters, therefore, to reduce the complexity of the dispersion equation, the fields would be solved only for specific cases (i.e., the forward and backward traveling waves would have the same amplitude).

Keeping this in mind during the analysis performed in the next few chapters, the fields are matched across the interface to get the dispersion relation

$$\alpha tan(\alpha d) = \epsilon_r \beta. \tag{1.8}$$

Going back to analysis, the dispersion equation is multiplied by d, the thickness of the dielectric (metal thickness is assumed negligible in our frequency range of interest) [9], we get $(\alpha d)tan(\alpha d) = \epsilon_r \beta d$ and from the attenuation constant equations, we get $(\alpha d)^2 + (\beta d)^2 =$

$(\epsilon_r - 1)(k_0 d)^2$, which represents a circle. Therefore, we have two equations in two unknowns αd and βd. Pozar [2] solves these equations graphically and depending on the radius (i.e., on ϵ_r), there would be points where the two curves would intersect, implying a solution for αd and βd, as shown in fig. 1.6.

A similar procedure is adopted for TE_z case and instead of tangent function, the dispersion relation has a cotangent function. Pozar [2] plots the curves for TE_z mode in a similar fashion to TM_z mode as shown in fig. 1.7. From fig. 1.7, it can be seen that a the number of surface wave modes depend on the value of the substrate permittivity ϵ_r.

1.5 Frequency Dependent Permittivity

In the previous sections, a basic understanding of dispersion relation was constructed and some concerns were put forth. In this section, two more issues are addressed before going into details of a general dispersion relation in the next chapters. Until now, the waveguide has been loaded with a frequency independent permittivity. Therefore, the radius of the circle, traversed by the equation $(\alpha d)^2 + (\beta d)^2 = (\epsilon_r - 1)(k_0 d)^2$, i.e., $k_0 d \sqrt{(\epsilon_r - 1)}$, is a constant for a given value of ϵ_r. Depending on the value of ϵ_r, therefore, the circle intersects the tangent (cotangent in case of TE_z) at two or more points.

But when ϵ_r is a function of frequency ω, instead of a circle, there would be a family of circles intersects the tangent or cotangent curve at many points, similar to that obtained by Liu et al. [13] and Cory et al. [54]. This is depicted in fig. 1.8. It will therefore, be a
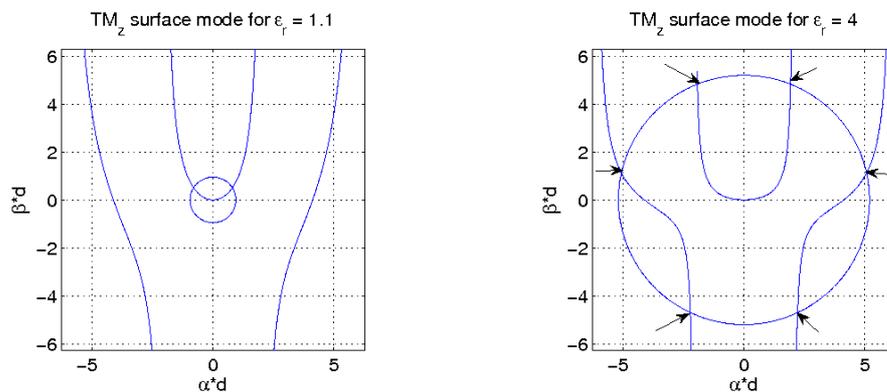
Fig. 1.6: TM_z surface wave modes for different values of ϵ_r.
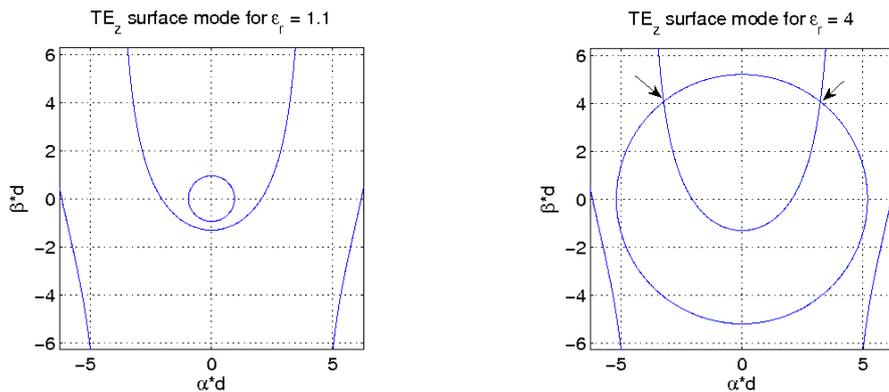
Fig. 1.7: TE_z surface wave modes for different values of ϵ_r.

difficult task to determine the surface modes for such an $\epsilon_r(\omega)$. Instead of plotting the two equations, the dispersion relation would be plotted for certain frequencies and the surface waves would be determined from such a curve. The situation would be worsened when ϵ_p is a tensor. For such scalar permittivities and tensors, therefore, plots of propagation constant against input frequency is an easier option [12, 29, 32]. Therefore, through the course of this work, the propagating wave numbers, corresponding to surface waves will be determined through root-searching algorithms, such as GA.

1.6 TM/TE Decomposition

For a general medium, Helmholtz equation (wave equation) is given by $\vec{\nabla} \times \vec{\nabla} \times \vec{A} = k_0^2 \epsilon \cdot \vec{A}$, where $\vec{A}$ is either of the electric or magnetic fields and ϵ can be a frequency independent/dependent scalar/tensor. For free space and for mediums with no free electric charges (i.e., $\vec{\nabla} \cdot \vec{A} = 0$), the equation reduces to

$$\nabla^2 \vec{A} + k_0^2 \epsilon \vec{A} = 0, \tag{1.9}$$

where $\vec{A}$ can be either electric or magnetic field. When this wave equation is solved for the field quantities, they are independent of each other. Therefore, all the field quantities cannot be determined from any two quantities. Pozar [2] and many other papers, such as Mok and Davis [15] and Bailey and Deshpande [55], did a TE/TM decomposition of
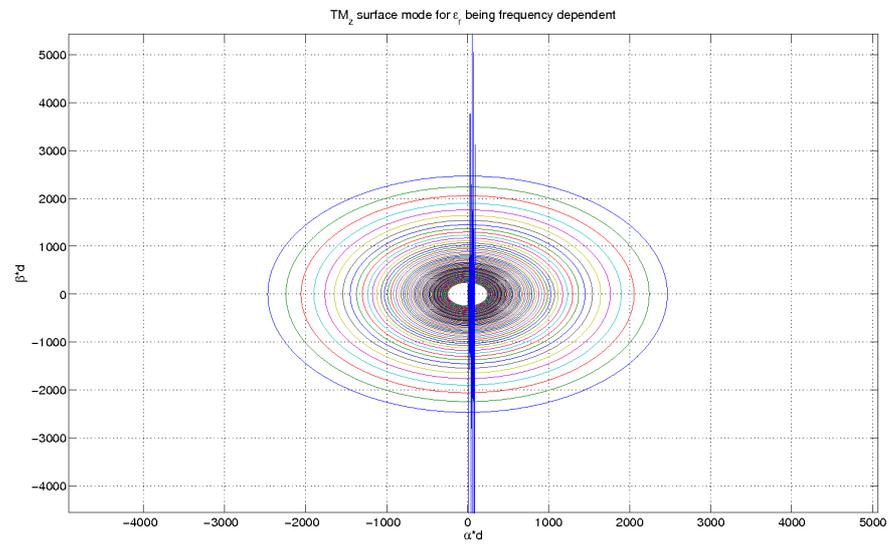
(a) Dispersion relation plot for TM_z mode for a frequency dependent permittivity.
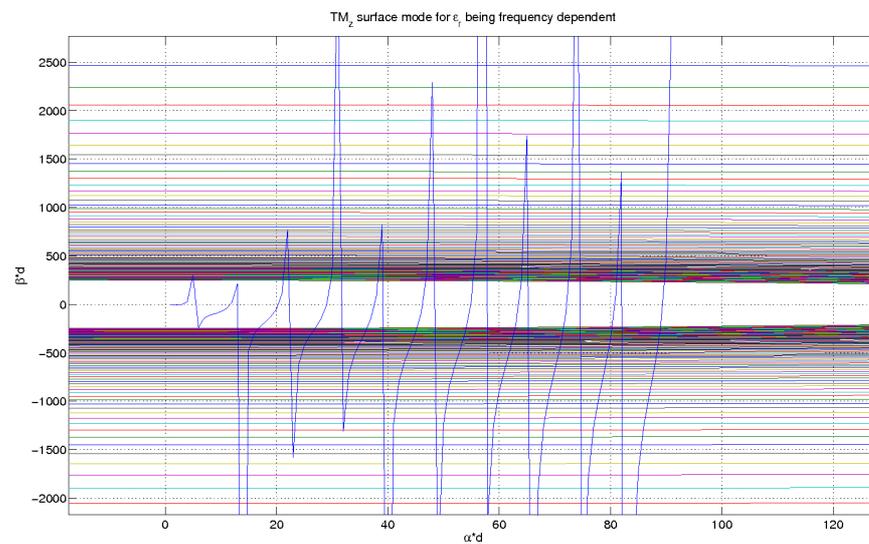
(b) Zoomed in plot to show the points of intersection.

Fig. 1.8: TM_z surface wave poles for ϵ_r being frequency dependent.

field quantities to circumvent these problems. To do such a decomposition, a propagation direction is assumed and the fields are solved for accordingly. Such a decomposition would therefore, be used whenever such degenerate forms occur [49]. Such a case will happen for the anisotropic case when $\vec{B}_0$ is parallel to the interface for Voigt geometry.

Also, the decomposition works only when ϵ_r is a scalar or it would even work for a uniaxial anisotropic permittivity tensor of the form

$$\epsilon = \begin{bmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{xx} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix}. \tag{1.10}$$

Even if the permittivity tensor is a diagonal matrix with different elements, this technique can work. The reason why this decomposition works for the permittivity configurations mentioned above is that when the wave equation is solved for such a permittivity, the field quantities are decoupled, so that a wave equation for any field can be worked out to get the dispersion relation. So, the TE/TM decomposition will be utilized for the cases where we get permittivity tensors which are uniaxial, biaxial, or even for the most simplest case, i.e., when $\vec{B}_0 = 0$ and $\vec{\nabla}p = 0$, which would yield an isotropic permittivity [54, 56].

But when the permittivity tensor has off-diagonal unequal elements, all the three x, y, and z components of the wave equation are needed to get a wave equation that has a single field variable. An example will make this point clear. Let ϵ be a tensor of the form

$$\tilde{\epsilon} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}. \tag{1.11}$$

The wave equation would be of the type

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \omega^2 \mu_0 \epsilon_0 \tilde{\epsilon} \cdot \vec{E}, \tag{1.12}$$

whose x, y, z components will be

$$\hat{x} \Rightarrow \frac{\partial^2}{\partial x \partial y}E_y - \frac{\partial^2}{\partial y^2}E_x + \frac{\partial^2}{\partial z \partial x}E_z - \frac{\partial^2}{\partial z^2}E_x = k_0^2(\epsilon_{xx}E_x + \epsilon_{xy}E_y$$
$$+\epsilon_{xz}E_z), \qquad (1.13)$$

$$\hat{y} \Rightarrow -(\frac{\partial^2}{\partial x^2}E_y - \frac{\partial^2}{\partial x \partial y}E_x - \frac{\partial^2}{\partial y \partial z}E_z + \frac{\partial^2}{\partial z^2}E_y) = k_0^2(\epsilon_{yx}E_x + \epsilon_{yy}E_y$$
$$+\epsilon_{yz}E_z), \qquad (1.14)$$

and

$$\hat{z} \Rightarrow -\frac{\partial^2}{\partial x^2}E_z + \frac{\partial^2}{\partial z \partial x}E_x - \frac{\partial^2}{\partial y^2}E_z + \frac{\partial^2}{\partial z \partial y}E_y = k_0^2(\epsilon_{zx}E_x + \epsilon_{zy}E_y$$
$$+\epsilon_{zz}E_z), \qquad (1.15)$$

where $k_0^2 = \omega^2 \epsilon_0 \mu_0$ is the free space wave number. For an isotropic material of relative permittivity ϵ_r on the other hand, the components would be

$$\hat{x} \Rightarrow \frac{\partial^2}{\partial x \partial y}E_y - \frac{\partial^2}{\partial y^2}E_x + \frac{\partial^2}{\partial z \partial x}E_z - \frac{\partial^2}{\partial z^2}E_x = k_0^2 \epsilon_r E_x, \qquad (1.16)$$

$$\hat{y} \Rightarrow -(\frac{\partial^2}{\partial x^2}E_y - \frac{\partial^2}{\partial x \partial y}E_x - \frac{\partial^2}{\partial y \partial z}E_z + \frac{\partial^2}{\partial z^2}E_y) = k_0^2 \epsilon_r E_y, \qquad (1.17)$$

$$\hat{z} \Rightarrow -\frac{\partial^2}{\partial x^2}E_z + \frac{\partial^2}{\partial z \partial x}E_x - \frac{\partial^2}{\partial y^2}E_z + \frac{\partial^2}{\partial z \partial y}E_y = k_0^2 \epsilon_r E_z. \qquad (1.18)$$

When looking for TM/TE modes, one needs to first ascertain the direction of propagation [2]. Let the direction of propagation be the z direction. Then for TM_z/TE_z decomposition, H_z or E_z respectively, would be zero. Therefore, all the transverse field components could then be written in terms of the longitudinal component (along the direction of propagation). As can be seen readily, for an isotropic material, because ϵ_r is a scalar, there is only one field component on the right hand side of the equation and therefore, deriving an equation of single field component would be easy. But when dealing with an anisotropic material, the right hand side contains all the three components, implying that the equations are coupled.

So, decomposition into TM_z/TE_z would not be as straightforward. Therefore, instead of using this technique, we will derive a general relation without assuming a TM/TE mode of propagation [12].

1.7   Spatial Dispersion

Spatial dispersion occurs when there are density perturbations in a medium [31, 32, 47, 48]. Macroscopically speaking, this effect is accounted for by wave vector dependence of permittivity $\tilde{\epsilon}_p$ and permeability. Since throughout the expose, we will be dealing with nonmagnetic materials, our main concern would be to get a permittivity tensor from the fluid equations, called the transport equations for semiconductors [32, 57–60]. Until now, we have only dealt with temporal dispersion, by the virtue of which total field at a point in time is given not only by the input given at that time but also at previous instants. So, for a linear and homogeneous material, one can write the constitutive relation between $\vec{E}$ and $\vec{D}$ as

$$\vec{D}(\vec{x};\omega) = \tilde{\epsilon}_p(\omega) \cdot \vec{E}(\vec{x};\omega), \tag{1.19}$$

where the quantities have been temporally Fourier transformed. In eq. (1.19), $\tilde{\epsilon}_p$ is a frequency dependent tensor [31]. Also, multiplication in temporal frequency domain implies convolution in time, so the above equation can be written in time as

$$\vec{D}(\vec{x};t) = \int_{-\infty}^{t} \epsilon_p(t - t') \cdot \vec{E}(\vec{x};t')dt'. \tag{1.20}$$

Equation (1.20) implies that there is nonlocal behavior in time in the sense that the displacement vector at a point $\vec{x}$ and time t depends on electric field not only at that point and time, but also on the previous times. In other words, there is a causal relation between $\vec{D}$ and $\vec{E}$. This nonlocality in time is what causes temporal dispersion, whereby after some finite time, a signal will be dispersed [47].

When spatial dispersion is also included in a medium's behavior, we encounter nonlocality both in time and space. The same constitutive relation between electric displacement

and field is then modified analogously to temporal dispersion [47, 48], to accommodate the spatial dispersion as

$$\vec{D}(\vec{x}; \omega) = \int \tilde{\epsilon}_p(\vec{x} - \vec{x}'; \omega) \cdot \vec{E}(\vec{x}'; \omega) d^3 x', \tag{1.21}$$

which when Fourier transformed spatially gives

$$\vec{D}(\vec{k}; \omega) = \tilde{\epsilon}_p(\vec{k}; \omega) \cdot \vec{E}(\vec{k}; \omega). \tag{1.22}$$

It is evident from the constitutive relation in spatial coordinates, that some assumptions need to be made on the density perturbations, so that $\tilde{\epsilon}_p$ is a function of $\vec{x} - \vec{x}'$. This can be achieved by assuming that the medium is translationally invariant in directions parallel to the interface, i.e., in y and z directions [32]. Also, by using the dielectric approximation, i.e., neglecting surface corrections to the tensor [32], the tensor can be assumed a function of $|x - x'|$. Note that $\vec{x}$ is the coordinate system and x is one of the three coordinate directions.

While doing spatial dispersion, the main aim would be to derive a permittivity tensor in spatial and temporal frequencies through the use of fluid equations [57, 58, 61] of the form

$$\epsilon_p(\vec{k}; \omega) = \epsilon_0 + \frac{\chi}{k^2 - \gamma^2(\omega)}, \tag{1.23}$$

such that when the tensor is inverse Fourier transformed in space, we get a Green's function for infinite space, i.e.,

$$\epsilon_p(\vec{x} - \vec{x}'; \omega) = \epsilon_0 \delta(\vec{x} - \vec{x}') + \frac{\chi}{2\pi} G_\gamma(\vec{x} - \vec{x}'). \tag{1.24}$$

$G_\gamma(\vec{x} - \vec{x}') = \frac{e^{j\gamma(\vec{x} - \vec{x}')}}{\vec{x} - \vec{x}'}$ is the Green's function for infinite space, χ is a constant and γ is a function of ω, the input frequency. The functional forms of these parameters will be given in Chapter 5, which will deal with spatial dispersion. The different terms in the equations will be defined in the chapter which deals with spatial dispersion. The electric field and Green's function can then be written as two-dimensional Fourier transforms, in the directions that

are infinite (i.e., y and z) as

$$\vec{E}(\vec{x};\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\vec{E}(k_y,k_z,x;\omega)e^{j(k_y y + k_z z)}dk_y dk_z, \qquad (1.25)$$

and

$$G_\gamma(|\vec{x}-\vec{x}'|) = \frac{j}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{w_\gamma}e^{j(k_y(y-y')+k_z(z-z')+w_\gamma|x-x'|)}dk_y dk_z, \qquad (1.26)$$

respectively, where the inverse Fourier transform of Green's function has been modified using the Weyl formula [23, 25, 47, 62] and $w_\gamma = \sqrt{(\gamma^2 - k_y^2 - k_z^2)}$.

There are other theories for derivation of a dispersion relation in a spatially nonlocal medium, such as Specular reflection method [32, 63]. This method will not be viable in the present context as it only applies for infinite mediums. More details about the derivation of dispersion relation through the Green's function technique will be analyzed in Chapter 5.

Chapter 2

Derivation of the General Dispersion Relation

As mentioned in the introductory chapter, a dispersion relation is an algebraic equation that relates the spatial wave number with the input signal frequency. It gives information about what modes can propagate in a given medium or different media separated by an interface [29, 32, 64–66]. As opposed to the full wave solutions, viz. MoM, FEM, FDTD, etc. [9, 10, 20], the dispersion relation method is found from the differential form of the wave equation using a defined spatial and temporal variation function for $\vec{E}$ and $\vec{H}$ fields called the guided wave functions or solutions [16, 19, 21, 24, 53], converting the wave equation into an algebraic equation instead of solving an electromagnetic problem with assumed fictitious sources as done by the prior methods.

2.1   Field Solutions

Before starting the derivation for the most general case of a semiconductor material separated from free space by a metal interface, assumptions are first made on the form of the field solutions that are defined as surface waves. From the previous chapter, we can very well foresee that a surface wave solution should have a exponentially decaying nature along the direction perpendicular to the interface (i.e., x direction) [12, 28], in both the mediums as shown in fig. 2.1.

For the semiconductor material, we then have for $\vec{E}$

$$\vec{E}(x, y, z, t) = \vec{E}_0 e^{-\alpha x} e^{jk_y y + jk_z z - j\omega t}, \tag{2.1}$$

and

$$\vec{E}(x, y, z, t) = \vec{E}_1 e^{-\beta x} e^{jk_y y + jk_z z - j\omega t}, \tag{2.2}$$
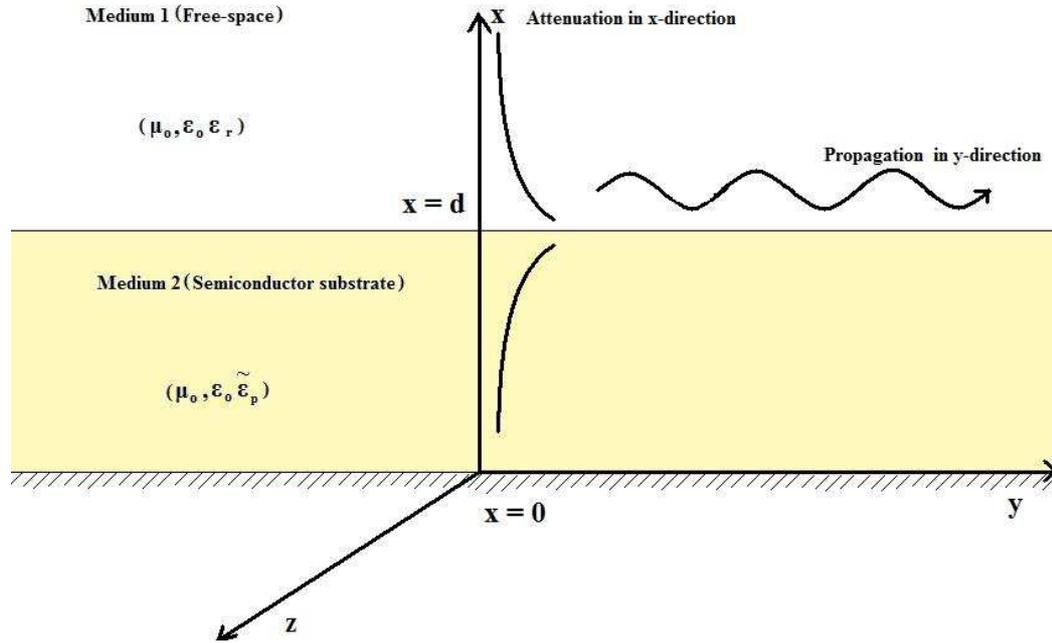
Fig. 2.1: Depiction of a surface wave mode attenuating exponentially perpendicular to and traveling along the interface.

for free space. The setup can understood more clearly from the fig. 2.2 [12].

The magnetic field will also have the same form, but since through Maxwell's curl equation (Faraday's Law), we can get $\vec{H}$ components if we have the $\vec{E}$ components, it is suffice to solve just for electric field quantities [29]. The above mentioned dependence will result in us solving an algebraic equation rather than a partial differential equation. Before doing that, the second curl equation should be made the same as the first one. This can be done by employing the momentum and continuity equations. This would make the curl equations as if the semiconductor medium was homogeneous, except for the fact that the permittivity would now be a tensor instead of a scalar [19,24]. Lets say that the permittivity tensor is given as

$$\tilde{\epsilon} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}. \tag{2.3}$$

This makes the curl equations (with the sinusoidal dependence) [16] as
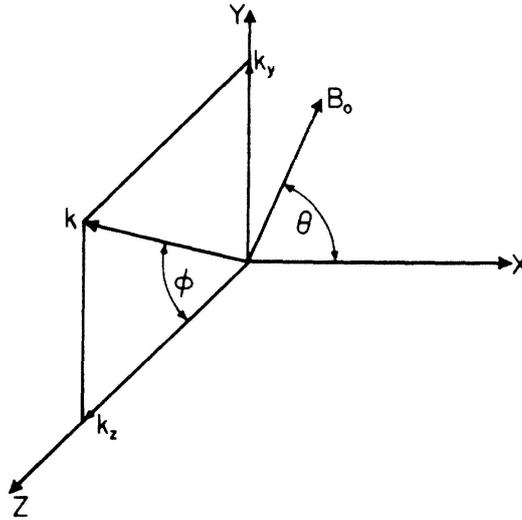
Fig. 2.2: Setup of the coordinate system for the problem.

$$\vec{\nabla} \times \vec{E}(x,y,z) \;\; = \;\; j\omega\mu_0\vec{H}(x,y,z), \tag{2.4}$$

$$\vec{\nabla} \times \vec{H}(x,y,z) \;\; = \;\; -j\omega\epsilon_0\tilde{\epsilon} \cdot \vec{E}(x,y,z), \tag{2.5}$$

where the semiconductor material has been assumed nonmagnetic. The wave equation then follows simply as

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \omega^2\mu_0\epsilon_0\tilde{\epsilon} \cdot \vec{E}. \tag{2.6}$$

2.2   Derivation of the Algebraic Form of Wave Equations and Field Solutions in Semiconductor

It should be noted that the semiconductor material is of thickness d along the x direction and extends infinitely in both y and z directions. Sign of the x dependent exponent indicates that α and β, the decay constant inside the semiconductor material and in free space respectively, should both be positive for a bonafide surface wave [12, 29]. Applying

the above mentioned dependence on the wave equation, we get

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\alpha & jk_y & jk_z \\ ik_y E_{0z} - ik_z E_{0y} & \alpha E_{0z} + ik_z E_{0x} & -\alpha E_{0y} - ik_y E_{0x} \end{bmatrix}, \quad (2.7)$$

where $\vec{E}_0$ are the constants for the x, y, and z components of the electric field. The following three algebraic equations result

$$\hat{x} \implies -jk_y \alpha E_{0y} + k_y^2 E_{0x} - ik_z \alpha E_{0z} + k_z^2 E_{0x} = k_0^2 (\epsilon_{xx} E_{0x} + \epsilon_{xy} E_{0y} + \epsilon_{xz} E_{0z}), \quad (2.8)$$

$$\hat{y} \implies -\alpha^2 E_{0y} - jk_y \alpha E_{0x} - k_z k_y E_{0z} + k_z^2 E_{0y} = k_0^2 (\epsilon_{yx} E_{0x} + \epsilon_{yy} E_{0y} + \epsilon_{yz} E_{0z}), \quad (2.9)$$

and

$$\hat{z} \implies -\alpha^2 E_{0z} - jk_z \alpha E_{0x} + k_y^2 E_{0z} - k_y k_z E_{0y} = k_0^2 (\epsilon_{zx} E_{0x} + \epsilon_{zy} E_{0y} + \epsilon_{zz} E_{0z}). \quad (2.10)$$

From the x component, we get

$$E_{0x} = \frac{(jk_y \alpha + k_0^2 \epsilon_{xy}) E_{0y} + (jk_z \alpha + k_0^2 \epsilon_{xz}) E_{0z}}{\kappa^2}, \quad (2.11)$$

where $\kappa^2 = k_y^2 + k_z^2 - k_0^2 \epsilon_{xx}$ for the ease of calculation.

Thus, we get two linear homogeneous equations for E_{0y} and E_{0z} as

$$A(\alpha) E_{0y} + B(\alpha) E_{0z} = 0, \quad (2.12)$$

$$C(\alpha) E_{0y} + D(\alpha) E_{0z} = 0, \quad (2.13)$$

where

$$A(\alpha) = (-\alpha^2 + k_z^2 - k_0^2 \epsilon_{yy})\kappa^2 - (-k_y^2 \alpha^2 + jk_y \alpha k_0^2 \epsilon_{xy} + jk_y \alpha k_0^2 \epsilon_{yx} + k_0^4 \epsilon_{yx} \epsilon_{xy}),$$

$$B(\alpha) = (-k_z k_y - k_0^2 \epsilon_{yz})\kappa^2 - (-k_y k_z \alpha^2 + jk_y \alpha k_0^2 \epsilon_{xz} + jk_z \alpha k_0^2 \epsilon_{yx} + k_0^4 \epsilon_{yx} \epsilon_{xz}),$$

and

$$C(\alpha) = (-k_y k_z - k_0^2 \epsilon_{zy})\kappa^2 - (-k_y k_z \alpha^2 + jk_z \alpha k_0^2 \epsilon_{xy} + jk_y \alpha k_0^2 \epsilon_{zx} + k_0^4 \epsilon_{zx}\epsilon_{xy}),$$

$$D(\alpha) = (-\alpha^2 + k_y^2 - k_0^2 \epsilon_{zz})\kappa^2 - (-k_z^2 \alpha^2 + jk_z \alpha k_0^2 \epsilon_{xz} + jk_z \alpha k_0^2 \epsilon_{zx} + k_0^4 \epsilon_{zx}\epsilon_{xz}).$$

Either of the above mentioned two linear algebraic eqs. (2.12) or (2.13) can be solved by assuming

$$E_{0y} = \sum_i F_i D(\alpha_i), \tag{2.14}$$

and

$$E_{0z} = -\sum_i F_i C(\alpha_i); \tag{2.15}$$

or

$$E_{0y} = \sum_i F_i B(\alpha_i), \tag{2.16}$$

and

$$E_{0z} = -\sum_i F_i A(\alpha_i). \tag{2.17}$$

The index i depends on the number of roots of the characteristic equation governing the wave equation inside the semiconductor material or in other words, the secular determinant of the two algebraic equations [2, 12, 49] mentioned above (in terms of the x dependence, viz. α). Mathematically speaking this would boil down to

$$\Delta(\alpha, \omega, k_y, k_z) = \begin{vmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{vmatrix}, \tag{2.18}$$

or simply

$$A(\alpha)D(\alpha) - B(\alpha)C(\alpha) = 0. \tag{2.19}$$

Equation (2.19) is a polynomial in α, the roots of which are the decay rates inside the semiconductor.

Writing the determinant more explicitly, for the most general case of ϵ_p, we get

$$[(-\alpha^2 + k_z^2 - k_0^2\epsilon_{yy})\kappa^2 - (-k_y^2\alpha^2 + jk_y\alpha k_0^2\epsilon_{xy} + jk_y\alpha k_0^2\epsilon_{yx} + k_0^4\epsilon_{yx}\epsilon_{xy})]$$

$$[(-\alpha^2 + k_y^2 - k_0^2\epsilon_{zz})\kappa^2 - (-k_z^2\alpha^2 + jk_z\alpha k_0^2\epsilon_{xz} + jk_z\alpha k_0^2\epsilon_{zx} + k_0^4\epsilon_{zx}\epsilon_{xz})] -$$

$$[(-k_z k_y - k_0^2\epsilon_{yz})\kappa^2 - (-k_y k_z\alpha^2 + jk_y\alpha k_0^2\epsilon_{xz} + jk_z\alpha k_0^2\epsilon_{yx} + k_0^4\epsilon_{yx}\epsilon_{xz})]$$

$$[(-k_y k_z - k_0^2\epsilon_{zy})\kappa^2 - (-k_y k_z\alpha^2 + jk_z\alpha k_0^2\epsilon_{xy} + jk_y\alpha k_0^2\epsilon_{zx} + k_0^4\epsilon_{zx}\epsilon_{xy})] = 0. \quad (2.20)$$

This will yield a polynomial equation

$$w_1\alpha^4 + w_2\alpha^3 + w_3\alpha^2 + w_4\alpha + w_5 = 0, \tag{2.21}$$

where ws, i.e., the coefficients of the equation will be a function of ω, k_y, and k_z, depending on the different assumptions taken into account [12, 29, 49, 50].

As seen from eqs. (2.20) or (2.21), the determinant is of fourth order in α, implying that there are four roots. The decay constants α and β should be positive and real (or complex with positive real part) to render the surface wave decaying away from the interface (according to the notation adopted). If this is not the case (i.e., if the roots are imaginary or complex), the surface wave would be traveling rather than decaying in nature or would be exponentially growing if the roots are negative [21, 28]. Also, if one of the roots is real and the other is imaginary, the surface wave would be a pseudo-surface wave (it would be traveling normal to the interface) [12]. Returning to the full solution inside the semiconductor material is then be given by (assuming the first of the two solutions mentioned above),

$$E_y = \sum_i F_i D(\alpha_i) e^{-\alpha_i x} e^{i(k_y y + k_z z - \omega t)}, \tag{2.22}$$

$$E_z = -\sum_i F_i C(\alpha_i) e^{-\alpha_i x} e^{i(k_y y + k_z z - \omega t)}, \tag{2.23}$$

and

$$E_x = \sum_i \frac{(jk_y\alpha_i + k_0^2\epsilon_{xy})F_i D(\alpha_i) - (jk_z\alpha_i + k_0^2\epsilon_{xz})F_i C(\alpha_i)}{\kappa^2} e^{-\alpha_i x} e^{j(k_y y + k_z z - \omega t)}, \quad (2.24)$$

using eq. (2.11).

2.3 Algebraic Field Solutions in Free Space

In free space, the general solution is assumed as

$$\vec{E} = \vec{E}_1 e^{-\beta x} e^{j(k_y y + k_z z - \omega t)}, \tag{2.25}$$

as mentioned above. We are looking for solutions which are positive or complex with positive real part. The wave equation in free space is well known and is given by [2, 8, 19, 24]

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \omega^2 \mu_0 \epsilon_0 \epsilon_r \vec{E}. \tag{2.26}$$

Since free space is an isotropic medium with no free charges [12, 24, 32], there would be no cross terms as in the eq. (2.26) and the characteristic equation would be of second order. Therefore, eq. (2.26) can be simplified using the mentioned field solution for free space as

$$\nabla^2 \vec{E} + k_0^2 \epsilon_r \vec{E} = 0, \tag{2.27}$$

$$\Rightarrow (\beta^2 - k_y^2 - k_z^2 + k_0^2 \epsilon_r)\vec{E} = 0, \tag{2.28}$$

where $k_0^2 = \omega^2 \mu_0 \epsilon_0$ is the free space wave number and $\epsilon_r = 1$ for free space and equal to some other constant value, if a semi-infinite isotropic dielectric is assumed for $x > d$. The vector wave equation can be written in component form as in the case of the substrate, but since all the field quantities will have the same wave equation form, we just need this form of the equation in order to get the roots for β, which are

$$\beta_1 = \sqrt{(k_y^2 + k_z^2 - k_0^2 \epsilon_r)}, \tag{2.29}$$

$$\beta_2 = -\sqrt{(k_y^2 + k_z^2 - k_0^2 \epsilon_r)}. \tag{2.30}$$

Out of the two roots from eqs. (2.29) and (2.30), the one with the positive sign would be picked as a valid solution, since we need the fields to be finite as $x \to \infty$, which is not

the case with the negative root (β_2) [2, 12]. In other words, at very large values of x, the negative solution would grow exponentially, rendering it to be a nonphysical solution. As we now are left with a single root in free space, $\beta_1 = \beta$ from now on and it means the positive root of the wave equation.

The field quantities in free space are related through the divergence equation of $\vec{E}$, i.e.,

$$\vec{\nabla} \cdot \vec{E} = 0, \tag{2.31}$$

$$\Rightarrow (-\hat{x}\beta + \hat{y}jk_y + \hat{z}jk_z) \cdot (\hat{x}E_{1x} + \hat{y}E_{1y} + \hat{z}E_{1z}) = 0, \tag{2.32}$$

$$\Rightarrow -\beta E_{1x} + jk_y E_{1y} + jk_z E_{1z} = 0, \tag{2.33}$$

$$\Rightarrow E_{1x} = \frac{jk_y E_{1y} + jk_z E_{1z}}{\beta}, \tag{2.34}$$

since there are no free charges in free space. Also, it has been been assumed that free space is a dielectric of relative permittivity ϵ_r (which can be assumed 1 for vacuum).

To get a generalized dispersion relation, adequate number of boundary conditions must be applied such that we can relate the decay rate in semiconductor (α) to that of free space (β). Boundary conditions will depend on the number of unknowns. In the most general case given above, one would need six boundary conditions for six unknowns, namely $F_i s$, E_{1y}, and E_{1z}, where is would be 4 in count. To get a dispersion equation that is unique, for the given number of unknowns, we have tangential electric field condition (at the interface and the ground plane) and the continuity of tangential magnetic field at the interface, which gives us a total of six boundary conditions. This will ensure that our dispersion relation uniquely describes the problem at hand [10, 24, 52].

2.4 Matching of Fields

Applying the boundary conditions, i.e., tangential electric field on the ground plane is equal to zero, since it is infinite in extent in y and z directions [2] and continuous at the

interface, we get

$$E_{y1} \mid_{x=0} \Rightarrow F_1 D(\alpha_1) + F_2 D(\alpha_2) + F_3 D(\alpha_3) + F_4 D(\alpha_4) \quad = \quad 0, \qquad (2.35)$$

$$E_{z1} \mid_{x=0} \Rightarrow -(F_1 C(\alpha_1) + F_2 C(\alpha_2) + F_3 C(\alpha_3) + F_4 C(\alpha_4)) \quad = \quad 0, \qquad (2.36)$$

and

$$E_{y1} = E_{y2} \mid_{x=d} \Rightarrow F_1 D(\alpha_1) e^{-\alpha_1 d} + F_2 D(\alpha_2) e^{-\alpha_2 d} + F_3 D(\alpha_3) e^{-\alpha_3 d}$$
$$+ F_4 D(\alpha_4) e^{-\alpha_4 d} = E_{1y} e^{-\beta d}, \qquad (2.37)$$

$$E_{z1} = E_{z2} \mid_{x=d} \Rightarrow -(F_1 C(\alpha_1) e^{-\alpha_1 d} + F_2 C(\alpha_2) e^{-\alpha_2 d} + F_3 C(\alpha_3) e^{-\alpha_3 d}$$
$$+ F_4 C(\alpha_4) e^{-\alpha_4 d}) = E_{1z} e^{-\beta d}. \qquad (2.38)$$

Till now, four of the six mentioned conditions have been applied. Before applying the magnetic field boundary condition, we need to derive the magnetic field quantities in both the media in terms of the already derived $\vec{E}$ fields. For both the mediums, through Faraday's Law of Electromagnetic induction [2, 21, 24], i.e., $\vec{\nabla} \times \vec{E} = j\omega\mu_0 \vec{H}$, the tangential magnetic field quantities can be easily derived.

Firstly, we need a general expression for $\vec{H}$ in both semiconductor and free space, which can be written as

$$\vec{H}_1(x, y, z, t) = \vec{H}_0 e^{-\alpha x} e^{-j(k_y y + k_z z - \omega t)}, \qquad (2.39)$$

and

$$\vec{H}_2(x, y, z, t) = \vec{H}_1 e^{-\alpha x} e^{-j(k_y y + k_z z - \omega t)}, \qquad (2.40)$$

respectively. The field solution will be a sum of individual solutions, depending on the number of roots of the wave equation [12, 29, 49]. The tangential components of $\vec{H}$ are H_y and H_z, respectively. In terms of $\vec{E}$, these can be written in differential form as

$$-\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) \quad = \quad -\mu_0 \frac{\partial H_y}{\partial t}, \qquad (2.41)$$

and

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu_0 \frac{\partial H_z}{\partial t}, \tag{2.42}$$

or in algebraic form (using the mentioned general expressions) as

$$\alpha E_{0z} + ik_z E_{0x} = i\omega\mu_0 H_{0y}, \tag{2.43}$$

$$-\alpha E_{0y} - ik_y E_{0x} = i\omega\mu_0 H_{0z}, \tag{2.44}$$

or simply

$$H_{0y} = \frac{\alpha E_{0z} + ik_z E_{0x}}{i\omega\mu_0}, \tag{2.45}$$

$$H_{0z} = \frac{-\alpha E_{0y} - ik_y E_{0x}}{i\omega\mu_0}, \tag{2.46}$$

for semiconductor medium, and

$$\beta E_{1z} + ik_z E_{1x} = i\omega\mu_0 H_{1y}, \tag{2.47}$$

$$-\beta E_{1y} - ik_y E_{1x} = i\omega\mu_0 H_{1z}, \tag{2.48}$$

or

$$H_{1y} = \frac{\beta E_{1z} + ik_z E_{1x}}{i\omega\mu_0}, \tag{2.49}$$

$$H_{1z} = \frac{-\beta E_{1y} - ik_y E_{1x}}{i\omega\mu_0}, \tag{2.50}$$

in free space. Finally, the E_0s are substituted from the previously derived equations to get the final dispersion relation, which is an equation in F_is and $\vec{E}_1$s only [12, 28]. Therefore,

the tangential components of $\vec{H}_1$ are found to be

$$H_{0y} = \frac{1}{j\omega\mu_0} \sum_i -\alpha_i F_i C(\alpha_i) + \frac{jk_z}{\kappa^2} [(jk_y\alpha_i + k_0^2\epsilon_{xy})F_i D(\alpha_i)$$
$$-(jk_z\alpha_i + k_0^2\epsilon_{xz})F_i C(\alpha_i)], \qquad (2.51)$$

and

$$H_{0z} = \frac{1}{j\omega\mu_0} \sum_i -\alpha_i F_i D(\alpha_i) - \frac{jk_y}{\kappa^2} [(jk_y\alpha_i + k_0^2\epsilon_{xy})F_i D(\alpha_i)$$
$$-(jk_z\alpha_i + k_0^2\epsilon_{xz})F_i C(\alpha_i)], \qquad (2.52)$$

so that the full solution looks like

$$H_{y1}(x,y,z,t) = \sum_i H_{0y}e^{-\alpha_i x}e^{j(k_y y + k_z z - \omega t)}, \qquad (2.53)$$

$$H_{z1}(x,y,z,t) = \sum_i H_{0z}e^{-\alpha_i x}e^{j(k_y y + k_z z - \omega t)}, \qquad (2.54)$$

where H_{0y} and H_{0z} have been derived above and are functions of α, that is why they have been included inside the sum [12]. Once H_y and H_z have been derived, H_x can easily be determined, but since we have adequate number of boundary conditions, we do not need the continuity of normal magnetic field. This condition will come in handy while deriving a dispersion equation for the spatial dispersion case [31, 47]. To match the fields across the interface, we need the field values in free space. Since we have the E field expressions, we can similarly derive the H fields as

$$H_{1y} = \frac{1}{j\omega\mu_0}[\beta E_{1z} + \frac{jk_z}{\beta}(jk_y E_{1y} + jk_z E_{1z})], \qquad (2.55)$$

$$H_{1z} = \frac{-1}{j\omega\mu_0}[\beta E_{1y} + \frac{jk_y}{\beta}(jk_y E_{1y} + jk_z E_{1z})], \qquad (2.56)$$

so that the full solution in free space can written as

$$H_{y2}(x,y,z,t) = H_{1y}e^{-\beta x}e^{j(k_y y + k_z z - \omega t)}, \qquad (2.57)$$

and

$$H_{z2}(x,y,z,t) \;=\; H_{1z}e^{-\beta x}e^{j(k_yy+k_zz-\omega t)}. \tag{2.58}$$

The summation sign has not been put for free space solutions, since we need the fields to decay and become finite at very large distances [2], which reduces the number of solutions in free space to just one root. Also, the notation H_{1z} denotes the field coefficient value in free space, whereas H_{z2} denotes the full solution. Similarly, E_{y1}, E_{z1}, E_{y2}, and E_{z2} are the expressions for electric fields in medium 1 (semiconductor) and medium 2 (free space), respectively, that depend on x, y, and z coordinates. Also, the wave numbers parallel to the interface should be the same in both the media (according to Snel's law) [21, 24]. Therefore, they drop out or get cancelled when the fields are matched at the interface or equal to zero on the ground plane.

Having found the respective H fields in the two mediums, we can match them at the interface as

$$
\begin{aligned}
H_{y1} \;=\; H_{y2}\mid_{x=d} \Rightarrow & \frac{1}{j\omega\mu_0}\sum_i[-\alpha_iF_iC(\alpha_i) + \frac{jk_z}{\kappa^2}[(jk_y\alpha_i + k_0^2\epsilon_{xy})F_iD(\alpha_i) \\
& -(jk_z\alpha_i + k_0^2\epsilon_{xz})F_iC(\alpha_i)]]e^{-\alpha_id} = \frac{1}{j\omega\mu_0}[\beta E_{1z} + \frac{jk_z}{\beta}(jk_yE_{1y} + jk_zE_{1z})]e^{-\beta d},
\end{aligned}
\tag{2.59}
$$

$$
\begin{aligned}
H_{z1} \;=\; H_{z2}\mid_{x=d} \Rightarrow & \frac{1}{j\omega\mu_0}\sum_i[-\alpha_iF_iD(\alpha_i) - \frac{jk_y}{\kappa^2}[(jk_y\alpha_i + k_0^2\epsilon_{xy})F_iD(\alpha_i) \\
& -(jk_z\alpha_i + k_0^2\epsilon_{xz})F_iC(\alpha_i)]]e^{-\alpha_id} = \frac{-1}{j\omega\mu_0}[\beta E_{1y} + \frac{jk_y}{\beta}(jk_yE_{1y} + jk_zE_{1z})]e^{-\beta d},
\end{aligned}
\tag{2.60}
$$

where the common factor, $\frac{1}{j\omega\mu_0}$, can be cancelled.

2.5 Dispersion Relation for the Most General $\tilde{\epsilon}_p$

We have six linear algebraic equations in six unknowns, implying that we have a unique dispersion relation [52]. The determinant of the system of equations gives the required

dispersion relation. In matrix form, the equations can be written as

$$\begin{bmatrix} D(\alpha_1) & D(\alpha_2) & D(\alpha_3) & D(\alpha_4) & 0 & 0 \\ -C(\alpha_1) & -C(\alpha_2) & -C(\alpha_3) & -C(\alpha_4) & 0 & 0 \\ D_1 & D_2 & D_3 & D_4 & -e^{-\beta d} & 0 \\ -C_1 & -C_2 & -C_3 & -C_4 & 0 & -e^{-\beta d} \\ M_1 & M_2 & M_3 & M_4 & \frac{k_y k_z}{\beta}e^{-\beta d} & \frac{k_z^2-\beta^2}{\beta}e^{-\beta d} \\ N_1 & N_2 & N_3 & N_4 & \frac{\beta^2-k_y^2}{\beta}e^{-\beta d} & \frac{-k_y k_z}{\beta}e^{-\beta d} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ E_{1y} \\ E_{1z} \end{bmatrix} = 0, \quad (2.61)$$

where

$$\begin{aligned} D_1 &= D(\alpha_1)e^{-\alpha_1 d}, \\ D_2 &= D(\alpha_2)e^{-\alpha_2 d}, \\ D_3 &= D(\alpha_3)e^{-\alpha_3 d}, \\ D_4 &= D(\alpha_4)e^{-\alpha_4 d}, \end{aligned} \qquad (2.62)$$

$$\begin{aligned} C_1 &= C(\alpha_1)e^{-\alpha_1 d}, \\ C_2 &= C(\alpha_2)e^{-\alpha_2 d}, \\ C_3 &= C(\alpha_3)e^{-\alpha_3 d}, \\ C_4 &= C(\alpha_4)e^{-\alpha_4 d}, \end{aligned} \qquad (2.63)$$

$$M_1 = [-\alpha_1 C(\alpha_1) + \frac{jk_z}{\kappa^2}((jk_y\alpha_1 + k_0^2\epsilon_{xy})D(\alpha_1) - (jk_z\alpha_1 + k_0^2\epsilon_{xz})C(\alpha_1))]e^{-\alpha_1 d},$$

$$M_2 = [-\alpha_2 C(\alpha_2) + \frac{jk_z}{\kappa^2}((jk_y\alpha_2 + k_0^2\epsilon_{xy})D(\alpha_2) - (jk_z\alpha_2 + k_0^2\epsilon_{xz})C(\alpha_2))]e^{-\alpha_2 d},$$

$$M_3 = [-\alpha_3 C(\alpha_3) + \frac{jk_z}{\kappa^2}((jk_y\alpha_3 + k_0^2\epsilon_{xy})D(\alpha_3) - (jk_z\alpha_3 + k_0^2\epsilon_{xz})C(\alpha_3))]e^{-\alpha_3 d},$$

$$M_4 = [-\alpha_4 C(\alpha_4) + \frac{jk_z}{\kappa^2}((jk_y\alpha_4 + k_0^2\epsilon_{xy})D(\alpha_4) - (jk_z\alpha_4 + k_0^2\epsilon_{xz})C(\alpha_4))]e^{-\alpha_4 d},$$

$$(2.64)$$

and

$$
\begin{aligned}
N_1 &= [-\alpha_1 D(\alpha_1) - \frac{jk_y}{\kappa^2}((jk_y\alpha_1 + k_0^2\epsilon_{xy})D(\alpha_1) - (jk_z\alpha_1 + k_0^2\epsilon_{xz})C(\alpha_1))]e^{-\alpha_1 d}, \\
N_2 &= [-\alpha_2 D(\alpha_2) - \frac{jk_y}{\kappa^2}((jk_y\alpha_2 + k_0^2\epsilon_{xy})D(\alpha_2) - (jk_z\alpha_2 + k_0^2\epsilon_{xz})C(\alpha_2))]e^{-\alpha_2 d}, \\
N_3 &= [-\alpha_3 D(\alpha_3) - \frac{jk_y}{\kappa^2}((jk_y\alpha_3 + k_0^2\epsilon_{xy})D(\alpha_3) - (jk_z\alpha_3 + k_0^2\epsilon_{xz})C(\alpha_3))]e^{-\alpha_3 d}, \\
N_4 &= [-\alpha_4 D(\alpha_4) - \frac{jk_y}{\kappa^2}((jk_y\alpha_4 + k_0^2\epsilon_{xy})D(\alpha_4) - (jk_z\alpha_4 + k_0^2\epsilon_{xz})C(\alpha_4))]e^{-\alpha_4 d}.
\end{aligned}
$$
(2.65)

Although we have a relation between α and β, the matrix form of the general dispersion relation (eq. (2.61)) is very complex to solve numerically [28]. We need to find those k_y and k_z wave numbers that will satisfy the transcendental equation formed by taking the secular determinant of eq. (2.61). Since we already have the fourth order algebraic equation in semiconductor medium and a second order one in free space, the coefficients of which are functions of k_y, k_z, and ω, the above matrix will be reduced to a transcendental equation that will yield what wave numbers k_y and k_z satisfy the equation [12, 29, 32, 49]. Also, the above dispersion equation is valid only when spatial dispersion or density perturbation is neglected.

When spatial dispersion or pressure term $\vec{\nabla}p$ is included, the fluid momentum equation would include the $\vec{\nabla}p$ term [32, 64, 65], which would make the tensor a complicated function of temporal and spatial frequencies in different directions. As mentioned in the first chapter, the wave equations inside the substrate would be integro-differential in nature [31, 32, 47] and some operations will be needed to get to a differential equation. Before doing that, one would need to have a model of the permittivity such that the inverse Fourier transform of of the permittivity gives a function in the form of Green's function [23, 25–27] for infinite space. Once, the microscopic perturbations have been handled through the fluid momentum and continuity equations, macroscopic boundary conditions can be applied in order to get a surface wave dispersion relation [32]. It would become more clear when spatial dispersion is considered in detail while deriving $\tilde{\epsilon}_p$.

Chapter 3

Isotropic Semiconductor Medium and Derivation of Scalar Permittivity

3.1 Scalar Permittivity with Damping and Pressure Neglected

A semiconductor medium can be rendered isotropic in the absence of a steady magnetic field [64, 65]. As we will find through the course of the chapter, in absence of an external steady magnetic field, the cross terms in the momentum equation vanish and the medium becomes an isotropic material, the only difference being that the permittivity would be a frequency dependent one [12, 64]. Apart from the Maxwell's curl equations, we need momentum, continuity (if the material is spatially dispersive), and the current density ($\vec{J}$) equations, which can be written in general as [64–66]

$$\rho_s(\frac{\partial}{\partial t} + \vec{U}_s \cdot \vec{\nabla})\vec{U}_s = Q_s(\vec{E} + \vec{U}_s \times \vec{B}) - \vec{\nabla}P_s + \rho_s \sum_r \nu_{sr}(\vec{U}_r - \vec{U}_s), \quad (3.1)$$

$$\frac{\partial N_s}{\partial t} + \vec{\nabla} \cdot (N_s\vec{U}_s) = P - L, \quad (3.2)$$

$$\vec{J} = \sum_s N_s q_s \vec{U}_s, \quad (3.3)$$

for species s. By species, we mean the charge carriers, i.e., either holes or electrons.

In the above equations, N_s, ρ_s, $\vec{U}_s$, $\vec{U}_r$, P_s, Q_s, and ν_{sr} stand for number density per m^3, mass density, flow velocity of species s and r, pressure, total charge on species s, and collision frequency, respectively. P and L are production and loss mechanisms in the semiconductor which can be assumed to be zero or some value depending on time scales. Also,

$$\rho_s = m_s N_s, \quad (3.4)$$

and

$$Q_s = q_s N_s, \tag{3.5}$$

are the mass density and total charge respectively on species s. In eqs. (3.4) and (3.5), m_s is the effective mass of either electron or hole and q_s is the charge on the species. It is important to mention the two Maxwell's curl equations again so as to easily understand why the fluid equations are needed after all. They are

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{3.6}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}. \tag{3.7}$$

3.2 Perturbation Theory

Equations (3.1) and (3.2) cannot be solved analytically, as they are nonlinear in nature [64, 66]. To make them feasible to solve so as to extract a permittivity tensor $\tilde{\epsilon}_p$, the equations are linearized through the perturbation technique, in which every field quantity as well as number density, flow velocity, etc., are split into large DC unperturbed values and small AC perturbations, that depend on time [64–66], i.e.,

$$Z = Z_0 + Z_1 e^{-j\omega t}. \tag{3.8}$$

Doing this operation on eqs. (3.1), (3.2), and (3.6), would give us zeroth and a first order equations. All the nonlinear terms (second order and above) can then be neglected and the $\vec{U}_s \cdot \vec{\nabla} \vec{U}_s$ term is also neglected [66], firstly owing to the fact that there can be no Doppler effects inside such a small device and secondly since the bulk flow, i.e., $\vec{U}_0$, is zero. Therefore, the zeroth order perturbations are,

$$\frac{\partial N_0}{\partial t} = 0, \tag{3.9}$$

$$\Rightarrow N_0 = constant, \tag{3.10}$$

and

$$0 \;=\; 0, \tag{3.11}$$

assuming $\vec{U}_{0s}$ and $\vec{E}_0$ to be zero, i.e., no bulk flows and no background electric field [65].

The first and higher order perturbations for a given species i can be written as [64,66]

$$\frac{\partial N_{i1}}{\partial t} + \vec{\nabla} \cdot (N_0 + N_{i1})\vec{U}_{i1} \;=\; 0, \tag{3.12}$$

$$m_i(N_0 + N_{i1})(\frac{\partial}{\partial t} + \vec{U}_{i1} \cdot \vec{\nabla})\vec{U}_{i1} \;=\; q_i(N_0 + N_{i1})(\vec{E}_1 + \vec{U}_{i1} \times (\vec{B}_0 + \vec{B}_1))$$

$$-\vec{\nabla}p_i + m_i(N_0 + N_{i1})\sum_j \nu_{ij}(\vec{U}_{j1} - \vec{U}_{i1}), \tag{3.13}$$

$$\Rightarrow -j\omega m_i N_0 \vec{U}_{i1} \;=\; q_i N_0 \vec{E}_1, \tag{3.14}$$

$$\nabla \times \vec{E}_1 \;=\; j\omega \vec{B}_1, \tag{3.15}$$

$$\nabla \times \vec{H}_1 \;=\; \vec{J}_1 - j\omega\epsilon_0 \vec{E}_1, \tag{3.16}$$

which simplify to

$$-j\omega N_{i1} + \vec{\nabla} \cdot (N_0 \vec{U}_{i1}) \;=\; 0, \tag{3.17}$$

$$\vec{U}_{i1} \;=\; \frac{jq_i}{m_i\omega}\vec{E}_1. \tag{3.18}$$

Note that the constitutive relations, viz.

$$\vec{D} \;=\; \epsilon_0 \vec{E}, \tag{3.19}$$

$$\vec{B} \;=\; \mu_0 \vec{H}, \tag{3.20}$$

have been used and it has been assumed that the material is nonmagnetic and the relative permittivity (ϵ_r) is 1. The collision frequency has also been neglected for ease of explanation and as mentioned, the material has isotropic behavior due to absence of an external magnetic field $\vec{B}_0$. Now that we have a relation between the perturbed flow velocity and electric field, using the relation $\vec{J} = \sum Nq\vec{U}$, we can apply perturbation theory and get the required

relation between $\vec{J}$ and $\vec{E}$, noting that the background velocity $\vec{U}_0$ is zero [66]. Therefore,

$$\vec{J}_1 = N_0 q(\vec{u}_h - \vec{u}_e). \tag{3.21}$$

Equation (3.21) is valid for the case when the semiconductor is quasi-neutral (i.e., the background density for holes and electrons is equal) [12, 64], which in turn implies that there is no background DC potential or electric field. Also, $\vec{u}_h$ and $\vec{u}_e$ denote the perturbed hole and electron flow velocities and the negative sign in front of the electron velocity is because of the negative charge on the electron. From eq. (3.18) derived above, we get

$$\vec{u}_h = \frac{jq}{m_h \omega} \vec{E}_1, \tag{3.22}$$

$$\vec{u}_e = -\frac{jq}{m_e \omega} \vec{E}_1, \tag{3.23}$$

for holes and electrons, respectively. Substituting eqs. (3.22) and (3.23) in eq. (3.21), we get

$$\vec{J}_1 = N_0 q(\frac{jq}{\omega m_h} + \frac{jq}{\omega m_e})\vec{E}_1. \tag{3.24}$$

Equation (3.16) then becomes

$$\vec{\nabla} \times \vec{H}_1 = N_0 q(\frac{jq}{\omega m_h} + \frac{jq}{\omega m_e})\vec{E}_1 - j\omega\epsilon_0\vec{E}_1, \tag{3.25}$$

$$\Rightarrow \vec{\nabla} \times \vec{H}_1 = (\frac{jN_0 q^2}{\omega m_h} + \frac{jN_0 q^2}{\omega m_e} - j\omega\epsilon_0)\vec{E}_1, \tag{3.26}$$

which simplifies to give

$$\vec{\nabla} \times \vec{H}_1 = -j\omega\epsilon_0(1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{ph}^2}{\omega^2})\vec{E}_1, \tag{3.27}$$

where ω_{pe} and ω_{ph} are the respective plasma frequencies for electrons and holes. The isotropic permittivity can then be written as [65, 66]

$$\epsilon_p = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{ph}^2}{\omega^2}, \tag{3.28}$$

or in matrix form as

$$\epsilon_p(\omega) = \begin{bmatrix} 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{ph}^2}{\omega^2} & 0 & 0 \\ 0 & 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{ph}^2}{\omega^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{ph}^2}{\omega^2} \end{bmatrix}. \qquad (3.29)$$

Unlike ϵ_0 and ϵ_r, ϵ_p depends on frequency throughout the frequency spectrum of interest (although ϵ_r starts depending on frequency after a certain cutoff).

3.3 Derivation with Finite Damping

The derivation of semiconductor permittivity with finite damping can be done by including the collision term in momentum equation. Therefore, the first order perturbation in momentum equation for electrons and holes is given by [64–66]

$$-j\omega m_e N_0 \vec{u}_e \;\; = \;\; -q N_0 \vec{E}_1 + m_e N_0 \nu_{eh}(\vec{u}_h - \vec{u}_e), \qquad (3.30)$$

$$-j\omega m_h N_0 \vec{u}_h \;\; = \;\; q N_0 \vec{E}_1 + m_h N_0 \nu_{he}(\vec{u}_e - \vec{u}_h), \qquad (3.31)$$

where $\vec{u}_e$ and $\vec{u}_h$ are the small amplitude AC perturbations in electron and hole flows. Simplifying eqs. (3.30) and (3.31), we get

$$\vec{u}_e \;\; = \;\; -\frac{jq}{\omega m_e}\vec{E}_1 + \frac{j\nu_{eh}}{\omega}(\vec{u}_h - \vec{u}_e), \qquad (3.32)$$

$$\vec{u}_h \;\; = \;\; \frac{jq}{\omega m_h}\vec{E}_1 + \frac{j\nu_{he}}{\omega}(\vec{u}_e - \vec{u}_h). \qquad (3.33)$$

Finally, the two equations are subtracted to get a relation between $\vec{u}_h - \vec{u}_e$ and $\vec{E}_1$ which is

$$\vec{u}_h - \vec{u}_e = \frac{\frac{jq}{\omega m_e} + \frac{jq}{\omega m_h}}{1 + \frac{j\nu_{eh}}{\omega} + \frac{j\nu_{he}}{\omega}}\vec{E}_1. \qquad (3.34)$$

Substituting eq. (3.34) back into eq. (3.21)

$$\vec{J}_1 \;\; = \;\; N_0 q \frac{\frac{jq}{\omega m_e} + \frac{jq}{\omega m_h}}{1 + \frac{j\nu_{eh}}{\omega} + \frac{j\nu_{he}}{\omega}}\vec{E}_1, \qquad (3.35)$$

and then into eq. (3.16)

$$\vec{\nabla} \times \vec{H}_1 \;=\; -j\omega\epsilon_0 \left(1 - \left(\frac{\frac{N_0 q^2}{\omega^2 m_e \epsilon_0} + \frac{N_0 q^2}{\omega^2 m_h \epsilon_0}}{1 + \frac{j\nu_{eh}}{\omega} + \frac{j\nu_{he}}{\omega}}\right)\right)\vec{E}_1, \tag{3.36}$$

we get

$$\nabla \times \vec{H}_1 \;=\; -j\omega\epsilon_0 \left(1 - \left(\frac{\frac{\omega_{pe}^2}{\omega^2} + \frac{\omega_{ph}^2}{\omega^2}}{1 + \frac{j\nu_{eh}}{\omega} + \frac{j\nu_{he}}{\omega}}\right)\right)\vec{E}_1, \tag{3.37}$$

and the required permittivity ϵ_p as

$$\epsilon_p \;=\; 1 - \left(\frac{\frac{\omega_{pe}^2}{\omega^2} + \frac{\omega_{ph}^2}{\omega^2}}{1 + \frac{j\nu_{eh}}{\omega} + \frac{j\nu_{he}}{\omega}}\right). \tag{3.38}$$

In tensor form, the substrate permittivity can be written as [65]

$$\epsilon_p(\omega) = \begin{bmatrix} 1 - \left(\frac{\frac{\omega_{pe}^2}{\omega^2} + \frac{\omega_{ph}^2}{\omega^2}}{1 + \frac{j\nu_{eh}}{\omega} + \frac{j\nu_{he}}{\omega}}\right) & 0 & 0 \\[2em] 0 & 1 - \left(\frac{\frac{\omega_{pe}^2}{\omega^2} + \frac{\omega_{ph}^2}{\omega^2}}{1 + \frac{j\nu_{eh}}{\omega} + \frac{j\nu_{he}}{\omega}}\right) & 0 \\[2em] 0 & 0 & 1 - \left(\frac{\frac{\omega_{pe}^2}{\omega^2} + \frac{\omega_{ph}^2}{\omega^2}}{1 + \frac{j\nu_{eh}}{\omega} + \frac{j\nu_{he}}{\omega}}\right) \end{bmatrix}. \tag{3.39}$$

ϵ_p, even though a scalar, has been written in tensor form for ease of transition into the anisotropic tensor case. The plot for ϵ_p is shown in fig. 3.1, for negligible (ϵ_{p1}) and finite (ϵ_{p2}) damping.

3.4 Dispersion Relation and Results for Isotropic Behavior (TE_y case)

Having derived the scalar permittivity for a semiconductor medium in absence of a steady magnetic field, we can now move on to the derivation of dispersion relation. As mentioned in the introduction, for scalar permittivity, we can assume a TM or TE mode, because when the propagation direction is fixed to a certain coordinate, one of the three wave equations will become completely independent of the other two [15, 54]. So, the only avenue left would be to break down the fields into certain modes, which leads to such
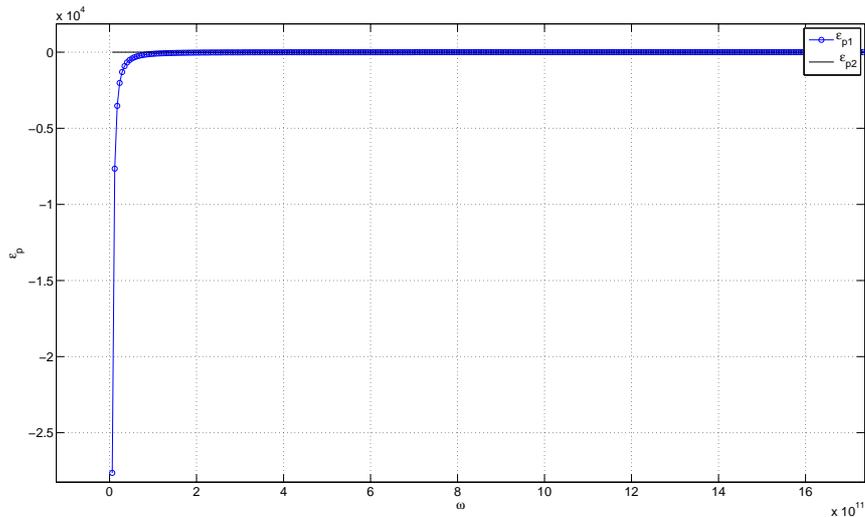
Fig. 3.1: Isotropic permittivity of the semiconductor substrate.

solutions. This is even true for a diagonal permittivity tensor with unequal elements, such as a uniaxial or biaxial substrate [14, 39]. The wave equation in terms of electric field is

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} \quad = \quad k_0^2 \epsilon_p(\omega) \vec{E}, \tag{3.40}$$

and with the field distribution as

$$\vec{E}_1(x, y, z; \omega) \quad = \quad \vec{E}_0 e^{-\alpha x} e^{j(k_y y + k_z z - \omega t)}, \tag{3.41}$$

$$\vec{E}_2(x, y, z; \omega) \quad = \quad \vec{E}_1 e^{-\beta x} e^{j(k_y y + k_z z - \omega t)}, \tag{3.42}$$

in the semiconductor substrate and in free space, respectively. We still do not know the number of roots α and β. This will become evident only while solving the wave equation in the two mediums. We are looking for wave solutions that propagate in y direction and decay in x direction, with no z dependence [2, 35]. The wave equation in semiconductor

substrate for a TE_y mode is written in matrix form as

$$
\begin{bmatrix}
\hat{x} & \hat{y} & \hat{z} \\
-\alpha & jk_y & 0 \\
jk_y E_{0z} & \alpha E_{0z} & -\alpha E_{0y} - jk_y E_{0x}
\end{bmatrix}
= k_0^2 \epsilon_p(\omega)
\begin{bmatrix}
E_{0x} \\
E_{0y} \\
E_{0z}
\end{bmatrix},
\tag{3.43}
$$

which in component form is

$$
\hat{x} \Rightarrow jk_y(-\alpha E_{0y} - jk_y E_{0x}) = k_0^2 \epsilon_p E_{0x},
\tag{3.44}
$$

$$
\hat{y} \Rightarrow \alpha(-\alpha E_{0y} - jk_y E_{0x}) = k_0^2 \epsilon_p E_{0y},
\tag{3.45}
$$

$$
\hat{z} \Rightarrow -\alpha^2 E_{0z} + k_y^2 E_{0z} = k_0^2 \epsilon_p E_{0z}.
\tag{3.46}
$$

The form of the wave equations tells that there is some arbitrariness in the isotropic case when $\frac{\partial}{\partial z} = 0$. In other words, the E_z component (eq. (3.46)) is completely independent of the other two, which means that one would get a different solution for E_z and E_x, E_y combined. So, this arbitrariness forces a TM and TE solution, implying that the field quantities E_z, H_x, H_y would be solved as TE_y mode and H_z, E_x, E_y as TM_y mode [15, 35, 54, 55]. Note that when TM_y mode will be solved, it will imply $H_z = 0$ rather than $H_y = 0$. This weird notation comes in because of the cross terms (since $\vec{\nabla} \cdot \vec{E} \neq 0$ in the substrate) in the wave equation. TM_y just means that we are looking for transverse magnetic wave solutions propagating in y direction and vice-versa, as shown in fig. 3.2. From eq. (3.46), we get $\alpha^2 = k_y^2 - k_0^2 \epsilon_p$ as the two roots. E_{0z} is therefore of the form $\sum_i F_i$, where $i = 2$ for the present case. The total solution is given as

$$
E_{z1} = (F_1 e^{-\alpha_1 x} + F_2 e^{-\alpha_2 x}) e^{-j(k_y y - \omega t)}.
\tag{3.47}
$$

On the other side of the interface, i.e., in medium 2, we similarly get

$$
E_{z2} = E_{1z} e^{-\beta x} e^{-j(k_y y - \omega t)},
\tag{3.48}
$$
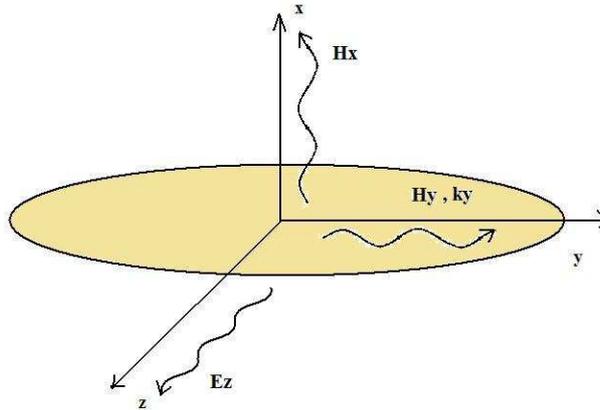
Fig. 3.2: Pictorial representation of a TE_y mode.

where the negative root is not considered, since we need the fields to decay to a finite value as $x \to \infty$. Applying the boundary condition on E_{z1} at $x = 0$, we get

$$E_{z1}|_{z=0} = 0 \Rightarrow F_1 = -F_2. \tag{3.49}$$

The solution looks like

$$E_{z1} = F_1(e^{\alpha_2 x} - e^{-\alpha_2 x})e^{-j(k_y y - \omega t)}, \tag{3.50}$$

since $\alpha_1 = -\alpha_2$. Simplifying eq. (3.50), we get

$$E_{z1} = 2F_1 sinh(\alpha_2 x)e^{-j(k_y y - \omega t)}, \tag{3.51}$$

$$\Rightarrow E_{z1} = F sinh(\alpha_2 x)e^{-j(k_y y - \omega t)}, \tag{3.52}$$

where the constant $2F_1$ has been replaced by F. Applying the continuity of E_z at $x = d$, we get

$$E_{z1} = E_{z2}|_{x=d} \Rightarrow F sinh(\alpha_2 d) = E_{1z}e^{-\beta d}. \tag{3.53}$$

Since we have two unknowns, we need another condition that relates F and E_{1z}. This condition is the continuity of H_y at the interface [12, 29], since the other tangential magnetic field component $H_z = 0$ (for TM_y mode). From eq. (3.15), we use the y component to get

$$H_{0y} = \frac{\alpha}{j\omega\mu_0}E_{0z}. \tag{3.54}$$

Since H_{0y} has α dependence in the coefficient, we will use $E_{0z} = \sum_i F_i$. Doing this and using $F_1 = -F_2$, along with $\alpha_1 = -\alpha_2$, we get after some algebra

$$H_{0y} = \frac{\alpha}{j\omega\mu_0}(F_1\alpha_1 + F_2\alpha_2), \tag{3.55}$$

$$\Rightarrow H_{y1} = \frac{1}{j\omega\mu_0}(F_1\alpha_1 e^{-\alpha_1 x} + F_2\alpha_2 e^{-\alpha_2 x})e^{-j(k_y y - \omega t)}. \tag{3.56}$$

Further simplification of eq. (3.56) yields

$$H_{y1} = \frac{-\alpha_2 F}{j\omega\mu_0}cosh(\alpha_2 x)e^{-j(k_y y - \omega t)}, \tag{3.57}$$

where some precaution should be taken when coefficients are functions of α.

A similar procedure for free space yields $H_{1y} = \frac{\beta}{j\omega\mu_0}E_{1z}$. Matching the two fields at the interface, we get the second relation between F and E_{1z} and finally the dispersion relation as

$$-\alpha_2 F cosh(\alpha_2 d) = \beta E_{1z}e^{-\beta d}, \tag{3.58}$$

$$\begin{bmatrix} sinh(\alpha_2 d) & -e^{-\beta d} \\ \alpha_2 cosh(\alpha_2 d) & \beta e^{-\beta d} \end{bmatrix} \begin{bmatrix} F \\ E_{1z} \end{bmatrix} = 0. \tag{3.59}$$

The determinant of the above matrix gives the required dispersion relation. As can be seen, the equation is a transcendental equation in k_y, since α and β are both functions of k_y. The above dispersion relation, i.e., eq. (3.59), was coded into MATLAB and the GA toolbox was used to minimize the relation using a range of frequencies. The sweep of frequencies was chosen so as to include the hole and electron resonances. The cases studied were effect

of collisions, effect of different background densities, and effect of different dimensions.

Figure 3.3 shows that when collisions are included in the the substrate permittivity, there is a distinct shift in the positions of the surface wave roots.

Having two semiconductor substrates with different background density results in different hole and electron plasma frequencies, which results in shifting in the dispersion curves for the two densities as shown in fig. 3.4. The size of the finite dimension (i.e., x direction) also plays a part in defining the number of surface wave modes that can exist in a slab. With $d_1 = .000762m$ and $d_2 = 10d_1$, for d_2 (thicker substrate), the modes increased by ten fold as seen in fig. 3.5.

It is important to note that the GA runs were done at a low resolution, so the function values would not go exactly to zero a low number. When the same code was run at a higher resolution, the surface modes reduced the value of the dispersion relation to a low enough number.

3.5 Dispersion Relation and Results for Isotropic Behavior (TM_y case)

For the TM_y mode, a similar procedure is undertaken as done for the TE_y mode. Writing the wave equation for the magnetic field, i.e., $\vec{\nabla} \times \vec{\nabla} \times \vec{H} = k_0^2 \tilde{\epsilon}_p \cdot \vec{H}$, we have in matrix form

$$
\begin{bmatrix}
\hat{x} & \hat{y} & \hat{z} \\
-\alpha & jk_y & 0 \\
jk_y H_{0z} & \alpha H_{0z} & -\alpha H_{0y} - jk_y H_{0x}
\end{bmatrix}
= k_0^2 \epsilon_p(\omega)
\begin{bmatrix}
H_{0x} \\
H_{0y} \\
H_{0z}
\end{bmatrix},
\tag{3.60}
$$

where the same field distribution $\vec{A}(x, y; \omega) = \vec{A}_0 e^{-\alpha x} e^{j(k_y y - \omega t)}$, has been used [16, 19, 21].

Therefore, the three components of the wave equation are

$$
\hat{x} \Rightarrow jk_y(-\alpha H_{0y} - jk_y H_{0x}) = k_0^2 \epsilon_p H_{0x},
\tag{3.61}
$$

$$
\hat{y} \Rightarrow \alpha(-\alpha H_{0y} - jk_y H_{0x}) = k_0^2 \epsilon_p H_{0y},
\tag{3.62}
$$

$$
\hat{z} \Rightarrow -\alpha^2 H_{0z} + k_y^2 H_{0z} = k_0^2 \epsilon_p H_{0z}.
\tag{3.63}
$$
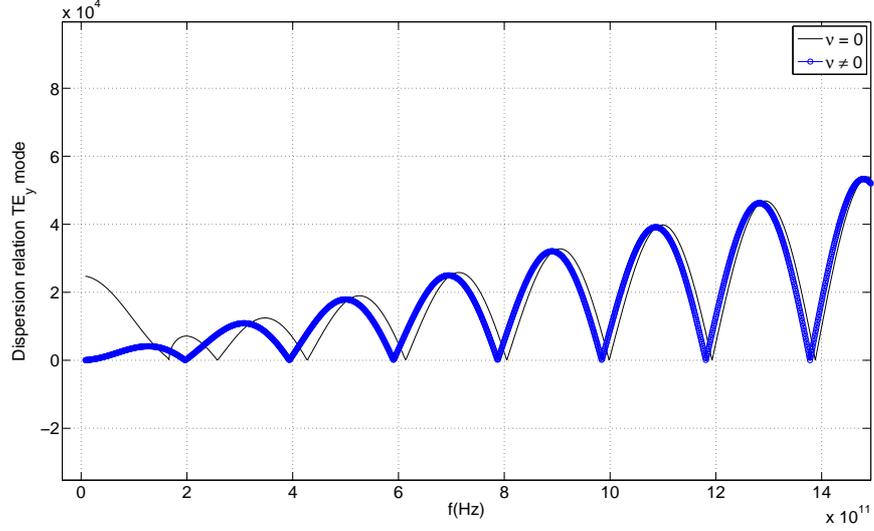
Fig. 3.3: TE_y mode dispersion relation for $\nu = 0$ and $\nu \neq 0$.

The same arbitrariness is found as in the TE_y case, since the z component is completely independent of the other two field components. Because of this, one has to resort to again solving for the longitudinal mode H_z (see fig. 3.6).

From eq. (3.63), we get $\alpha^2 = k_y^2 - k_0^2 \epsilon_p$ as the two roots. H_{0z} can then be written in the form $\sum_i F_i$, where $i = 2$ for the present case. The total solution is given as

$$H_{z1} = (F_1 e^{-\alpha_1 x} + F_2 e^{-\alpha_2 x}) e^{-j(k_y y - \omega t)}. \tag{3.64}$$

On the other side of the interface, we have

$$H_{z2} = H_{1z} e^{-\beta x} e^{-j(k_y y - \omega t)}, \tag{3.65}$$

where the negative root is not considered, since we need the fields to decay to a finite value as $x \to \infty$ [2], β being the solution to the wave equation $(\nabla^2 + k_0^2 \epsilon_r)\vec{H} = 0$ (since in free space, $\vec{\nabla} \cdot \vec{H} = 0$). We need to apply the boundary conditions on E_{tan1} at $x = 0$. Since $E_{z1} = 0$ for TE_y mode, we are left with E_{y1}, which can be found from the y component of
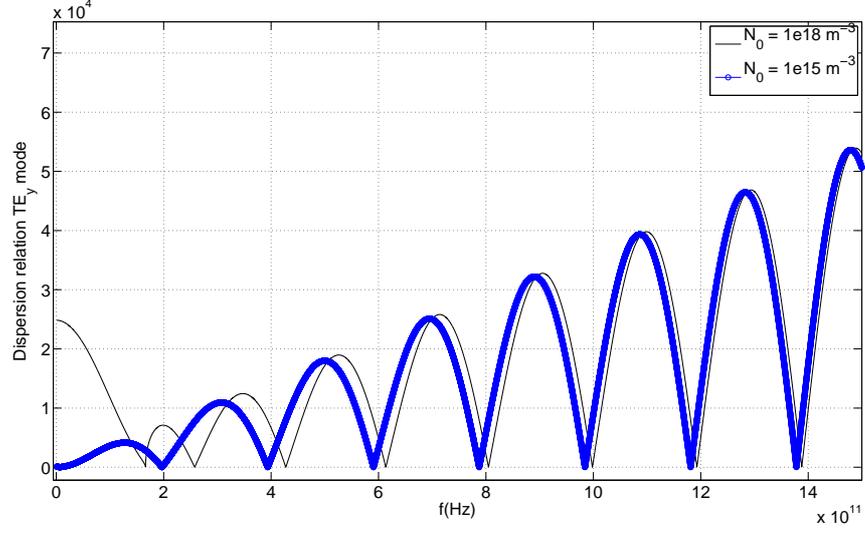
Fig. 3.4: TE_y mode dispersion relation for $N_0 = 10^{18}m^{-3}$ and $N_0 = 10^{15}m^{-3}$.

$\vec{\nabla} \times \vec{H} = -j\omega\epsilon_0\epsilon_p\vec{E}$ as

$$\vec{\nabla} \times \vec{H} = -j\omega\epsilon_0\epsilon_p\vec{E}, \tag{3.66}$$

$$\Rightarrow \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\alpha & jk_y & 0 \\ H_{0x} & H_{0y} & H_{0z} \end{bmatrix} = -j\omega\epsilon_0\epsilon_p\vec{E}_0, \tag{3.67}$$

$$\hat{y} \Rightarrow \alpha H_{0z} = -j\omega\epsilon_0\epsilon_p\vec{E}_{0y}, \tag{3.68}$$

$$\Rightarrow E_{0y} = \frac{-\alpha}{j\omega\epsilon_0\epsilon_p}H_{0z}. \tag{3.69}$$

Therefore, the total solution for E_{y1} is given by

$$E_{y1} = -\frac{1}{j\omega\epsilon_0\epsilon_p}(F_1\alpha_1 e^{-\alpha_1 x} + F_2\alpha_2 e^{-\alpha_2 x})e^{j(k_y y - \omega t)}, \tag{3.70}$$

since the coefficient E_{0y} is itself a function of α. $\alpha_1 = -\alpha_2$ simplifies eq. (3.70) into

$$E_{y1} = -\frac{1}{j\omega\epsilon_0\epsilon_p}(-F_1\alpha_2 e^{\alpha_2 x} + F_2\alpha_2 e^{-\alpha_2 x})e^{j(k_y y - \omega t)}. \tag{3.71}$$
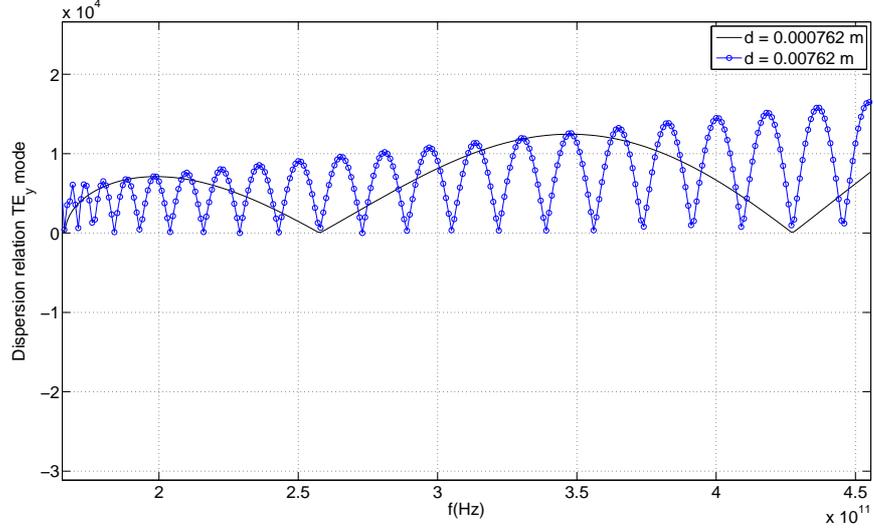
Fig. 3.5: TE_y mode dispersion relation for $d_1 = .000762m$ and $d_2 = 10d_1$.

Applying the boundary condition at the ground plane [2,28], i.e., $E_{y1}|_{x=0} = 0$ gives $F_1 = F_2$. Then, E_{y1} further simplifies to

$$E_{y1}(x,y;\omega) = -\frac{F_1\alpha_2}{j\omega\epsilon_0\epsilon_p}(-e^{\alpha_2 x} + e^{-\alpha_2 x})e^{j(k_y y - \omega t)}, \tag{3.72}$$

$$\Rightarrow E_{y1}(x,y;\omega) = \frac{F\alpha_2}{j\omega\epsilon_0\epsilon_p}sinh(\alpha_2 x)e^{j(k_y y - \omega t)}, \tag{3.73}$$

where $F = 2F_1$.

On the free space side, from eq. (3.65), using Maxwell's curl equation $\vec{\nabla} \times \vec{H}_2 = -j\omega\epsilon_0\epsilon_r\vec{E}_2$, E_{y2} is found using the y component to get

$$\beta H_{1z} = -j\omega\epsilon_0\epsilon_r E_{1y}, \tag{3.74}$$

$$\Rightarrow E_{1y} = -\frac{\beta}{j\omega\epsilon_0\epsilon_r}H_{1z}, \tag{3.75}$$

where $\epsilon_r = 1$ for free space and equal to any constant value for some arbitrary dielectric. The matching of tangential fields E_y across the interface $x = d$, gives

$$\frac{F}{\epsilon_p}\alpha_2 sinh(\alpha_2 d) = -\frac{\beta}{\epsilon_r}H_{1z}e^{-\beta d}, \tag{3.76}$$
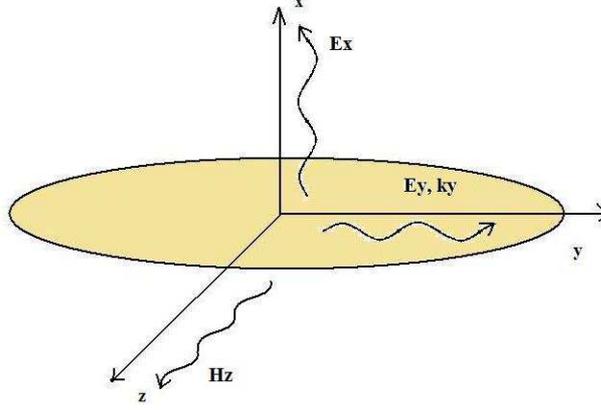
Fig. 3.6: A TM_y mode representation.

where the common factor, $\frac{1}{j\omega\epsilon_0}$, has been cancelled. Since we have two unknowns, viz. F and H_{1z}, we need another equation that relates F and H_{1z}. This is achieved by matching the tangential H fields at $x = d$ [12, 29] or the continuity of magnetic fields across the interface. This condition gives

$$F\cosh(\alpha_2 d) = H_{1z}e^{-\beta d}. \tag{3.77}$$

Then, we have the dispersion relation in matrix form as

$$\begin{bmatrix} \frac{\alpha_2}{\epsilon_p}\sinh(\alpha_2 d) & \frac{\beta}{\epsilon_r}e^{-\beta d} \\ \cosh(\alpha_2 d) & -e^{-\beta d} \end{bmatrix} \begin{bmatrix} F \\ H_{1z} \end{bmatrix} = 0. \tag{3.78}$$

The determinant of eq. (3.78) gives the required dispersion relation. The equation is a transcendental equation in k_y, with α and β substituted for in terms of k_y.

The same cases as TE_y mode were investigated and it was found that TM_y has the same behavior as a TE_y mode, since the electric and magnetic fields swap [2, 8], but their functional forms remain the same.

Chapter 4

Anisotropic Behavior and Derivation of Permittivity Tensor

Anisotropicity shows up in a semiconductor when there is a steady magnetic field ($\vec{B}_0$) [64, 65], as mentioned in the previous chapter. In presence of an external steady magnetic field, the cross terms in the momentum equation do not vanish and the medium becomes an anisotropic material, with the plasma permittivity (ϵ_p) being a frequency dependent tensor. The same procedure is applied to derive the tensor. Maxwell's curl equations, momentum, and continuity (if the material is spatially dispersive) fluid equations, are required, which can be written as

$$\rho_s(\frac{\partial}{\partial t} + \vec{U}_s \cdot \vec{\nabla})\vec{U}_s = Q_s(\vec{E} + \vec{U}_s \times \vec{B}) - \vec{\nabla}P_s + \rho_s \sum_r \nu_{sr}(\vec{U}_r - \vec{U}_s), \tag{4.1}$$

$$\frac{\partial N_s}{\partial t} + \vec{\nabla} \cdot (N_s\vec{U}_s) = P - L, \tag{4.2}$$

$$\vec{J} = \sum_s N_s q_s \vec{U}_s, \tag{4.3}$$

for species s. In eq. (4.3), $\vec{J}$ is the current density due to different species traveling at different velocities. All the terms have been explained in the previous chapter. The Maxwell's curl equations are the same as before, i.e.,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{4.4}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}. \tag{4.5}$$

These equations are linearized through the perturbation technique, in which every field quantity as well as number density, flow velocity, etc., are split into large DC unperturbed

values and small AC perturbations, that depend on time as [64–66]

$$Z = Z_0 + ze^{-j\omega t}. \tag{4.6}$$

Applying eq. (4.6) on eqs. (4.1), (4.2), (4.3), (4.4), and (4.5), we get the zeroth and first order perturbations. All the nonlinear terms (second order and above) can then be neglected along with $\vec{U}_s \cdot \vec{\nabla}\vec{U}_s$ as it is also a second order perturbation (since the background flow velocity is zero). Proceeding along these lines, the first order perturbations for electrons can be written as

$$-m_e N_0 j\omega \vec{u}_e \;=\; -qN_0(\vec{E}_1 + \vec{u}_e \times \vec{B}_0) + m_e N_0 \sum_j \nu_{eh}(\vec{u}_h - \vec{u}_e), \tag{4.7}$$

$$\nabla \times \vec{E}_1 \;=\; j\omega\vec{B}_1, \tag{4.8}$$

$$\nabla \times \vec{H}_1 \;=\; \vec{J}_1 - j\omega\epsilon_0\vec{E}_1. \tag{4.9}$$

Before going further with the derivation, it should be stated that the steady magnetic field is aligned at an arbitrary angle θ with the x axis, as shown in fig. 4.1, i.e.,

$$\vec{B}_0 = B_0(\hat{x}cos\theta + \hat{y}sin\theta). \tag{4.10}$$

Expanding the $\vec{u}_e \times \vec{B}_0$ term in eq. (4.7), which turns out to be the term that is responsible for anisotropy in absence of the $\vec{\nabla}p$ term, we get for the perturbed electron flow velocity

$$\vec{u}_e \times \vec{B}_0 = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_{ex} & u_{ey} & u_{ez} \\ B_0cos\theta & B_0sin\theta & 0 \end{bmatrix}, \tag{4.11}$$

$$\Rightarrow \vec{u}_e \times \vec{B}_0 = \hat{x}(-u_{ez}B_0sin\theta) - \hat{y}(-u_{ez}B_0cos\theta) + \hat{z}(u_{ex}B_0sin\theta$$

$$-u_{ey}B_0cos\theta). \tag{4.12}$$
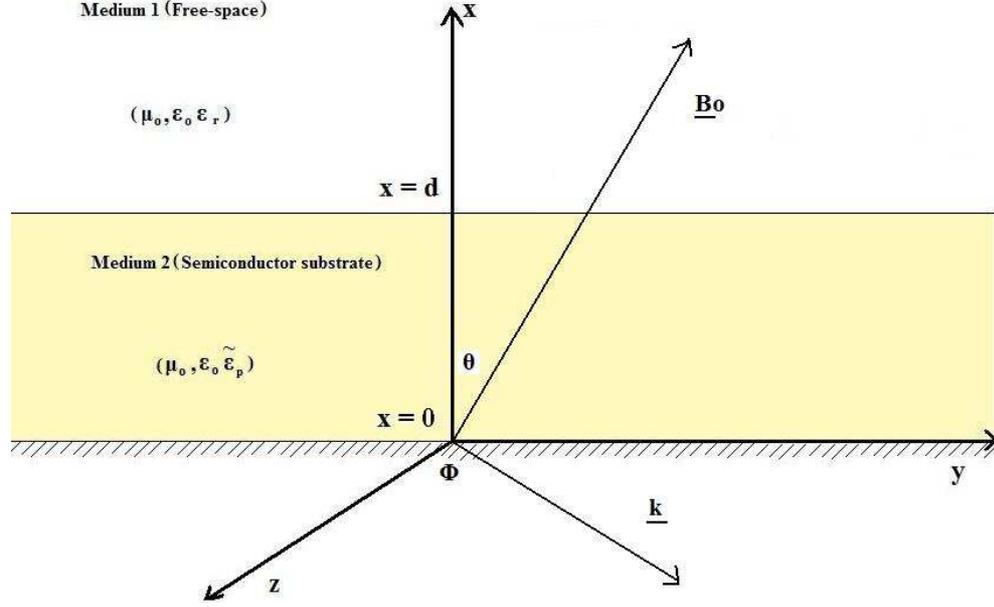
Fig. 4.1: Magnetic field and $\vec{k}$ vector orientations.

Equation (4.1) for electrons and holes then becomes [65],

$$-j\omega\vec{u}_e = -\frac{q}{m_e}(\vec{E} - \hat{x}u_{ez}B_0sin\theta + \hat{y}u_{ez}B_0cos\theta + \hat{z}(u_{ex}B_0sin\theta - u_{ey}B_0cos\theta))$$
$$+\nu_{eh}(\vec{u}_h - \vec{u}_e), \quad (4.13)$$

$$-j\omega\vec{u}_h = \frac{q}{m_h}(\vec{E} - \hat{x}u_{ez}B_0sin\theta + \hat{y}u_{ez}B_0cos\theta + \hat{z}(u_{ex}B_0sin\theta - u_{ey}B_0cos\theta))$$
$$+\nu_{he}(\vec{u}_e - \vec{u}_h). \quad (4.14)$$

The $\hat{x}$, $\hat{y}$, and $\hat{z}$ components for electrons using eq. (4.13) are

$$\hat{x} \Rightarrow \omega^2 u_{ex} = -\frac{j\omega q}{m_e}(E_x - u_{ez}B_0sin\theta) + j\omega\nu_{eh}(u_{hx} - u_{ex}), \quad (4.15)$$

$$\hat{y} \Rightarrow \omega^2 u_{ey} = -\frac{j\omega q}{m_e}(E_y + u_{ez}B_0cos\theta) + j\omega\nu_{eh}(u_{hy} - u_{ey}), \quad (4.16)$$

$$\hat{z} \Rightarrow \omega^2 u_{ez} = -\frac{j\omega q}{m_e}(E_z + u_{ex}B_0sin\theta - u_{ey}B_0cos\theta) + j\omega\nu_{eh}(u_{hz} - u_{ez}), \quad (4.17)$$

and for holes using eq. (4.14) are

$$\hat{x} \Rightarrow \omega^2 u_{hx} = \frac{j\omega q}{m_h}(E_x - u_{hz}B_0 sin\theta) + j\omega\nu_{he}(u_{ex} - u_{hx}), \quad (4.18)$$

$$\hat{y} \Rightarrow \omega^2 u_{hy} = \frac{j\omega q}{m_h}(E_y + u_{hz}B_0 cos\theta) + j\omega\nu_{he}(u_{ey} - u_{hy}), \quad (4.19)$$

$$\hat{z} \Rightarrow \omega^2 u_{hz} = \frac{j\omega q}{m_h}(E_z + u_{hx}B_0 sin\theta - u_{hy}B_0 cos\theta) + j\omega\nu_{he}(u_{ez} - u_{hz}), \quad (4.20)$$

where the two equations have been multiplied by $j\omega$ before the components were found for ease of calculation. Also, E denotes the first order perturbation and not the whole quantity having both DC and AC perturbation terms in it.

It is more feasible to write these in matrix form so that an inverse can be found readily or else finding a relation between $\vec{u}$ and $\vec{E}$ would be complex and time consuming [65]. The six equations can be written as

$$\begin{bmatrix} \omega^2 & 0 & \frac{-j\omega q B_0}{m_e}sin\theta \\ 0 & \omega^2 & \frac{j\omega q B_0}{m_e}cos\theta \\ \frac{j\omega q B_0}{m_e}sin\theta & \frac{-j\omega q B_0}{m_e}cos\theta & \omega^2 \end{bmatrix} \begin{bmatrix} u_{ex} \\ u_{ey} \\ u_{ez} \end{bmatrix} = -\frac{j\omega q}{m_e}\vec{E} + j\omega\nu_{eh}(\vec{u}_h - \vec{u}_e). (4.21)$$

Dividing throughout by ω^2, we finally get an equation that relates $\vec{u}_e$ to $\vec{E}$,

$$\begin{bmatrix} 1 & 0 & \frac{-j\Omega_{ce}}{\omega}sin\theta \\ 0 & 1 & \frac{j\Omega_{ce}}{\omega}cos\theta \\ \frac{j\Omega_{ce}}{\omega}sin\theta & \frac{-j\Omega_{ce}}{\omega}cos\theta & 1 \end{bmatrix} \begin{bmatrix} u_{ex} \\ u_{ey} \\ u_{ez} \end{bmatrix} = -\frac{jq}{\omega m_e}\vec{E} + \frac{j\nu_{eh}}{\omega}(\vec{u}_h - \vec{u}_e). \quad (4.22)$$

Similarily for holes, we get

$$\begin{bmatrix} \omega^2 & 0 & \frac{j\omega q B_0}{m_h}sin\theta \\ 0 & \omega^2 & \frac{-j\omega q B_0}{m_h}cos\theta \\ \frac{-j\omega q B_0}{m_h}sin\theta & \frac{j\omega q B_0}{m_h}cos\theta & \omega^2 \end{bmatrix} \begin{bmatrix} u_{hx} \\ u_{hy} \\ u_{hz} \end{bmatrix} = \frac{j\omega q}{m_h}\vec{E} + j\omega\nu_{he}(\vec{u}_e - \vec{u}_h). \quad (4.23)$$

Again, dividing throughout by ω^2, we get an equation that relates $\vec{u}_h$ to $\vec{E}$,

$$\begin{bmatrix} 1 & 0 & \frac{j\Omega_{ch}}{\omega}sin\theta \\ 0 & 1 & \frac{-j\Omega_{ch}}{\omega}cos\theta \\ \frac{-j\Omega_{ch}}{\omega}sin\theta & \frac{j\Omega_{ch}}{\omega}cos\theta & 1 \end{bmatrix} \begin{bmatrix} u_{hx} \\ u_{hy} \\ u_{hz} \end{bmatrix} = \frac{jq}{\omega m_h}\vec{E} + \frac{j\nu_{he}}{\omega}(\vec{u}_e - \vec{u}_h). \quad (4.24)$$

It is important to keep in mind that we need a relation between $\vec{u}_h - \vec{u}_e$ and $\vec{E}$, so that ultimately we arrive at an equation that relates $\vec{J}_1$ and $\vec{E}$ [64]. Writing eqs. (4.22) and (4.24) in a more compact form, we get

$$M_e \cdot \vec{u}_e = -\frac{jq}{\omega m_e}\vec{E} + \frac{j\nu_{eh}}{\omega}(\vec{u}_h - \vec{u}_e), \quad (4.25)$$

$$M_h \cdot \vec{u}_h = \frac{jq}{\omega m_h}\vec{E} + \frac{j\nu_{he}}{\omega}(\vec{u}_e - \vec{u}_h), \quad (4.26)$$

where

$$M_e = \begin{bmatrix} 1 & 0 & \frac{-j\Omega_{ce}}{\omega}sin\theta \\ 0 & 1 & \frac{j\Omega_{ce}}{\omega}cos\theta \\ \frac{j\Omega_{ce}}{\omega}sin\theta & \frac{-j\Omega_{ce}}{\omega}cos\theta & 1 \end{bmatrix}, \quad (4.27)$$

and

$$M_h = \begin{bmatrix} 1 & 0 & \frac{j\Omega_{ch}}{\omega}sin\theta \\ 0 & 1 & \frac{-j\Omega_{ch}}{\omega}cos\theta \\ \frac{-j\Omega_{ch}}{\omega}sin\theta & \frac{j\Omega_{ch}}{\omega}cos\theta & 1 \end{bmatrix}. \quad (4.28)$$

Since we need to subtract the electron and hole momentum equations, the two equations are multiplied by M_e^{-1} and M_h^{-1}, respectively, to give

$$\vec{u}_e = -\frac{jq}{\omega m_e}M_e^{-1} \cdot \vec{E} + \frac{j\nu_{eh}}{\omega}M_e^{-1} \cdot (\vec{u}_h - \vec{u}_e), \quad (4.29)$$

$$\vec{u}_h = \frac{jq}{\omega m_h}M_h^{-1} \cdot \vec{E} + \frac{j\nu_{he}}{\omega}M_h^{-1} \cdot (\vec{u}_e - \vec{u}_h), \quad (4.30)$$

and after subtracting

$$\vec{u}_e - \vec{u}_h = (-\frac{jq}{\omega m_e}M_e^{-1} - \frac{jq}{\omega m_h}M_h^{-1}) \cdot \vec{E} - (\frac{j\nu_{eh}}{\omega}M_e^{-1} + \frac{j\nu_{he}}{\omega}M_h^{-1}) \cdot (\vec{u}_e - \vec{u}_h), \quad (4.31)$$

and simplifying, we get

$$(I + \frac{j\nu_{eh}}{\omega}M_e^{-1} + \frac{j\nu_{he}}{\omega}M_h^{-1}) \cdot (\vec{u}_e - \vec{u}_h) = (-\frac{jq}{\omega m_e}M_e^{-1} - \frac{jq}{\omega m_h}M_h^{-1}) \cdot \vec{E}, \quad (4.32)$$

where I is the identity matrix. The required relation is then found to be

$$\vec{u}_e - \vec{u}_h = [I + \frac{j\nu_{eh}}{\omega}M_e^{-1} + \frac{j\nu_{he}}{\omega}M_h^{-1}]^{-1}(-\frac{jq}{\omega m_e}M_e^{-1} - \frac{jq}{\omega m_h}M_h^{-1}) \cdot \vec{E}. \quad (4.33)$$

Finally,

$$\vec{J} = N_0 q(\vec{u}_h - \vec{u}_e), \quad (4.34)$$

$$\vec{J} = -N_0 q[I + \frac{j\nu_{eh}}{\omega}M_e^{-1} + \frac{j\nu_{he}}{\omega}M_h^{-1}]^{-1}(-\frac{jq}{\omega m_e}M_e^{-1} - \frac{jq}{\omega m_h}M_h^{-1}) \cdot \vec{E}. \quad (4.35)$$

As can be seen from the amount of complexity in the equation, some simplification needs to be done. In the next section, it will be assumed that damping is negligible [65] and then including damping, the permittivity tensor will be found for different configurations of $\vec{B}_0$. The first case will be $\vec{B}_0$ perpendicular to the interface and next would be parallel to the interface. Special cases such as Voigt ($\vec{k} \perp \vec{B}_0$) and Faraday ($\vec{k} \parallel \vec{B}_0$) configurations [29, 49, 50] would then be adopted to make the analysis simpler.

4.1 Derivation of Permittivity Tensor with Negligible Damping

As mentioned in the previous section, the relation between $\vec{J}$ and $\vec{E}$, i.e., eq. (4.35), is too complex to be handled as it stands. Some simplifications need to be made, the first of which would be to neglect damping, i.e., $\nu \ll \omega$ [65]. Keeping this assumption in mind, the

electron and hole momentum equations become

$$\hat{x} \Rightarrow u_{ex} = -\frac{jq}{\omega m_e}(E_x - u_{ez}B_0 sin\theta), \tag{4.36}$$

$$\hat{y} \Rightarrow u_{ey} = -\frac{jq}{\omega m_e}(E_y + u_{ez}B_0 cos\theta), \tag{4.37}$$

$$\hat{z} \Rightarrow u_{ez} = -\frac{jq}{\omega m_e}(E_z + u_{ex}B_0 sin\theta - u_{ey}B_0 cos\theta), \tag{4.38}$$

$$\hat{x} \Rightarrow u_{hx} = \frac{jq}{\omega m_h}(E_x - u_{hz}B_0 sin\theta), \tag{4.39}$$

$$\hat{y} \Rightarrow u_{hy} = \frac{jq}{\omega m_h}(E_y + u_{hz}B_0 cos\theta), \tag{4.40}$$

$$\hat{z} \Rightarrow u_{hz} = \frac{jq}{\omega m_h}(E_z + u_{hx}B_0 sin\theta - u_{hy}B_0 cos\theta), \tag{4.41}$$

or in matrix form as

$$
\begin{bmatrix}
1 & 0 & \frac{-j\Omega_{ce}}{\omega}sin\theta \\
0 & 1 & \frac{j\Omega_{ce}}{\omega}cos\theta \\
\frac{j\Omega_{ce}}{\omega}sin\theta & \frac{-j\Omega_{ce}}{\omega}cos\theta & 1
\end{bmatrix}
\begin{bmatrix}
u_{ex} \\
u_{ey} \\
u_{ez}
\end{bmatrix}
= -\frac{jq}{\omega m_e}\vec{E}, \tag{4.42}
$$

$$
\begin{bmatrix}
1 & 0 & \frac{j\Omega_{ch}}{\omega}sin\theta \\
0 & 1 & \frac{-j\Omega_{ch}}{\omega}cos\theta \\
\frac{-j\Omega_{ch}}{\omega}sin\theta & \frac{j\Omega_{ch}}{\omega}cos\theta & 1
\end{bmatrix}
\begin{bmatrix}
u_{hx} \\
u_{hy} \\
u_{hz}
\end{bmatrix}
= \frac{jq}{\omega m_h}\vec{E}. \tag{4.43}
$$

In compact form eqs. (4.42) and (4.43) are written as

$$\vec{u}_e = -\frac{jq}{\omega m_e}M_e^{-1} \cdot \vec{E}, \tag{4.44}$$

$$\vec{u}_h = \frac{jq}{\omega m_h}M_h^{-1} \cdot \vec{E}, \tag{4.45}$$

where M_e^{-1} and M_h^{-1} have been stated previously. Already, deriving ϵ_p is now not as complex as it was previously. Subtracting eqs. (4.44) and (4.45), we get

$$\vec{u}_e - \vec{u}_h = -(\frac{jq}{\omega m_e}M_e^{-1} + \frac{jq}{\omega m_h}M_h^{-1}) \cdot \vec{E}, \tag{4.46}$$

$$\Rightarrow \vec{u}_h - \vec{u}_e = (\frac{jq}{\omega m_e}M_e^{-1} + \frac{jq}{\omega m_h}M_h^{-1}) \cdot \vec{E}. \tag{4.47}$$

Substituting eq. (4.47) back into $\vec{J} = N_0 q(\vec{u}_h - \vec{u}_e)$, we get

$$\vec{J} = (\frac{jN_0q^2}{\omega m_e} M_e^{-1} + \frac{jNq^2}{\omega m_h} M_h^{-1}) \cdot \vec{E}. \tag{4.48}$$

To get $\tilde{\epsilon}_p$, we need to find the inverse of the matrices M_e and M_h. They are

$$M_e^{-1} = \frac{1}{\Delta_e} \begin{bmatrix} 1 - X_e^2 cos^2\theta & -X_e^2 cos\theta sin\theta & jX_e sin\theta \\ -X_e^2 cos\theta sin\theta & 1 - X_e^2 sin^2\theta & -jX_e cos\theta \\ -jX_e sin\theta & jX_e cos\theta & 1 \end{bmatrix}, \tag{4.49}$$

$$M_h^{-1} = \frac{1}{\Delta_h} \begin{bmatrix} 1 - X_h^2 cos^2\theta & -X_h^2 cos\theta sin\theta & -jX_h sin\theta \\ -X_h^2 cos\theta sin\theta & 1 - X_h^2 sin^2\theta & jX_h cos\theta \\ jX_h sin\theta & -jX_h cos\theta & 1 \end{bmatrix}, \tag{4.50}$$

where $X_e = \frac{\Omega_{ce}}{\omega}$, $X_h = \frac{\Omega_{ch}}{\omega}$, and Δ_e, Δ_h are the respective determinants of M_e and M_h given by

$$\Delta_e = 1 - X_e^2, \tag{4.51}$$

$$\Delta_h = 1 - X_h^2. \tag{4.52}$$

Substituting eq. (4.48) into eq. (4.5), we get

$$\nabla \times \vec{H}_1 = \vec{J}_1 - j\omega\epsilon_0 \vec{E}_1, \tag{4.53}$$

$$\Rightarrow \nabla \times \vec{H} = -j\omega\epsilon_0 (I - \frac{\omega_{pe}^2}{\omega^2} M_e^{-1} - \frac{\omega_{ph}^2}{\omega^2} M_h^{-1}) \cdot \vec{E}, \tag{4.54}$$

where the field quantities $\vec{H}$ and $\vec{E}$ are first order perturbations. For the assumption $\nu \ll \omega$, we finally arrive at the permittivity tensor ϵ_p given by

$$\tilde{\epsilon}_p(\omega) = I - \frac{\omega_{pe}^2}{\omega^2} M_e^{-1} - \frac{\omega_{ph}^2}{\omega^2} M_h^{-1}, \tag{4.55}$$

where ω_{pe} and ω_{ph} are electron and hole plasma frequencies, respectively, as mentioned before.

4.2 $\vec{B}_0$ Perpendicular to the Interface with $\nu << \omega$

The dispersion relation for this case can be found if $\theta = 0°$ [64, 66] is substituted in the the permittivity matrix. In matrix form, the permittivity tensor $\tilde{\epsilon}_p(\omega)$ can be written as

$$\tilde{\epsilon}_p(\omega) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{\omega_{pe}^2}{\omega^2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{1-X_e^2} & -\frac{jX_e}{1-X_e^2} \\ 0 & \frac{jX_e}{1-X_e^2} & \frac{1}{1-X_e^2} \end{bmatrix} - \frac{\omega_{ph}^2}{\omega^2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{1-X_h^2} & \frac{jX_h}{1-X_h^2} \\ 0 & -\frac{jX_h}{1-X_h^2} & \frac{1}{1-X_h^2} \end{bmatrix}, \quad (4.56)$$

or after simplification as

$$\tilde{\epsilon}_p(\omega) = \begin{bmatrix} 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{ph}^2}{\omega^2} & 0 & 0 \\ 0 & 1 - \frac{\frac{\omega_{pe}^2}{\omega^2}}{1-X_e^2} - \frac{\frac{\omega_{ph}^2}{\omega^2}}{1-X_h^2} & jX_e \frac{\frac{\omega_{pe}^2}{\omega^2}}{1-X_e^2} - jX_h \frac{\frac{\omega_{ph}^2}{\omega^2}}{1-X_h^2} \\ 0 & -jX_e \frac{\frac{\omega_{pe}^2}{\omega^2}}{1-X_e^2} + jX_h \frac{\frac{\omega_{ph}^2}{\omega^2}}{1-X_h^2} & 1 - \frac{\frac{\omega_{pe}^2}{\omega^2}}{1-X_e^2} - \frac{\frac{\omega_{ph}^2}{\omega^2}}{1-X_h^2} \end{bmatrix}. \quad (4.57)$$

Equation (4.57) implies that the tensor has the form

$$\tilde{\epsilon}_p(\omega) = \begin{bmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & \epsilon_{yz} \\ 0 & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}. \quad (4.58)$$

Although some matrix elements, viz. ϵ_{yy} and ϵ_{zz}, are equal and some elements, viz. ϵ_{yz} and ϵ_{zy}, are the negative of each other, still they have been written as unequal elements since this will allow us to use the general dispersion relation derived in second chapter. The simplifications on the elements will be made once the final dispersion relation has been found. In the next two sections, dispersion relation would be found for Voigt ($\vec{k} \perp \vec{B}_0$) [12] and Faraday ($\vec{k} \parallel \vec{B}_0$) [29, 49] geometries, with $\vec{B}_0$ perpendicular to the interface.

4.2.1 Dispersion Relation for Voigt Geometry with $\nu << \omega$ and $\vec{B}_0$ Perpendicular to the Interface

For this geometry, since $\vec{k} \perp \vec{B}_0$ and $\vec{B}_0 = \hat{x}B_0$, the traveling wave numbers are k_y and k_z. We can pick either of the two mentioned wave numbers. So, let $\vec{k} = \hat{y}k_y$ and let $k_z = 0$, since we have rotational symmetry about B_0 as shown in fig. 4.2. This simplification has been made so that we finally arrive at a dispersion equation which will only be a function of k_y alone.

Inside the semiconductor substrate, solving the wave equation, we have the following coefficients of E_{0y} and E_{0z}:

$$A(\alpha) = (-\alpha^2 + k_z^2 - k_0^2\epsilon_{yy})\kappa^2 - (-k_y^2\alpha^2 + jk_y\alpha k_0^2\epsilon_{xy} + jk_y\alpha k_0^2\epsilon_{yx}$$
$$+k_0^4\epsilon_{yx}\epsilon_{xy}), \tag{4.59}$$

$$B(\alpha) = (-k_zk_y - k_0^2\epsilon_{yz})\kappa^2 - (-k_yk_z\alpha^2 + jk_y\alpha k_0^2\epsilon_{xz} + jk_z\alpha k_0^2\epsilon_{yx}$$
$$+k_0^4\epsilon_{yx}\epsilon_{xz}), \tag{4.60}$$

$$C(\alpha) = (-k_yk_z - k_0^2\epsilon_{zy})\kappa^2 - (-k_yk_z\alpha^2 + jk_z\alpha k_0^2\epsilon_{xy} + jk_y\alpha k_0^2\epsilon_{zx}$$
$$+k_0^4\epsilon_{zx}\epsilon_{xy}), \tag{4.61}$$

$$D(\alpha) = (-\alpha^2 + k_y^2 - k_0^2\epsilon_{zz})\kappa^2 - (-k_z^2\alpha^2 + jk_z\alpha k_0^2\epsilon_{xz} + jk_z\alpha k_0^2\epsilon_{zx}$$
$$+k_0^4\epsilon_{zx}\epsilon_{xz}), \tag{4.62}$$

which had already been derived in the second chapter dealing with the general dispersion relation. The permittivity tensor $\tilde{\epsilon}_p$ is of the form

$$\tilde{\epsilon}_p(\omega) = \begin{bmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & \epsilon_{yz} \\ 0 & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}. \tag{4.63}$$

We just need to simplify the equations according to the configuration so that the dispersion relation we finally get is specific to the configuration.
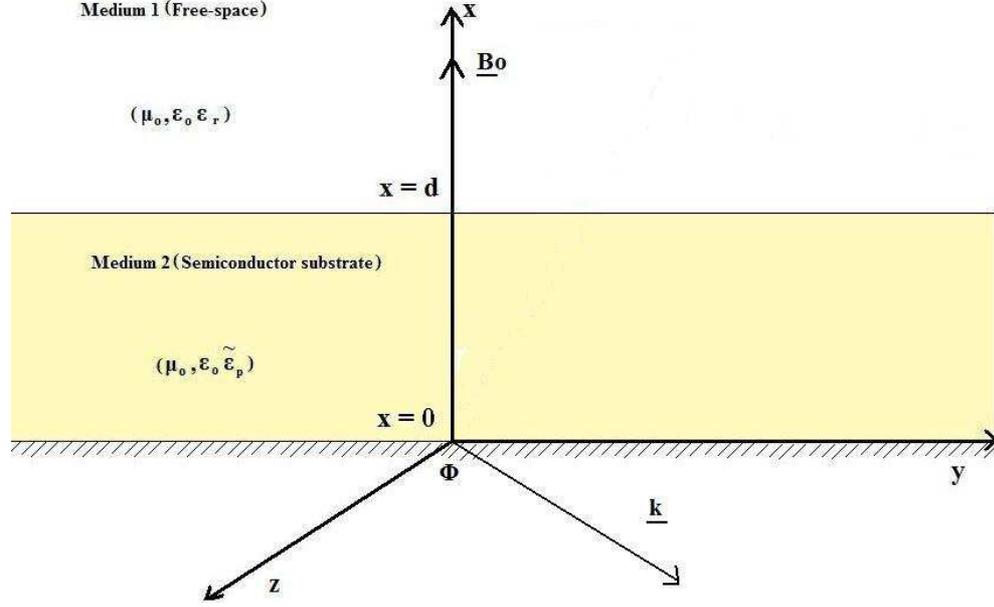
Fig. 4.2: $\vec{k}$ orientation for Voigt geometry around $\vec{B}_0$ along x direction.

For the present configuration, we have

$$k_z \;=\; 0, \tag{4.64}$$

$$\epsilon_{xy} \;=\; 0, \tag{4.65}$$

$$\epsilon_{xz} \;=\; 0, \tag{4.66}$$

$$\epsilon_{yx} \;=\; 0, \tag{4.67}$$

$$\epsilon_{zx} \;=\; 0, \tag{4.68}$$

so that A, B, C, D in eqs. (4.59)-(4.62) are reduced to

$$A(\alpha) \;=\; (-\alpha^2 - k_0^2 \epsilon_{yy})\kappa^2 - (-k_y^2 \alpha^2), \tag{4.69}$$

$$B \;=\; (-k_0^2 \epsilon_{yz})\kappa^2, \tag{4.70}$$

$$C \;=\; (-k_0^2 \epsilon_{zy})\kappa^2, \tag{4.71}$$

$$D(\alpha) \;=\; (-\alpha^2 + k_y^2 - k_0^2 \epsilon_{zz})\kappa^2, \tag{4.72}$$

where $\kappa = \sqrt{(k_y^2 - k_0^2 \epsilon_{xx})}$ and the α dependence on B and C has been removed, since they are no longer functions of α. We had the following interdependence between E_{0y} and E_{0z}:

$$A(\alpha)E_{0y} + BE_{0z} = 0, \tag{4.73}$$

$$CE_{0y} + D(\alpha)E_{0z} = 0, \tag{4.74}$$

and the other field component E_{0x} was related to E_{0y} and E_{0z} as

$$E_{0x} = \frac{jk_y \alpha E_{0y}}{\kappa^2}. \tag{4.75}$$

It is important to note here that although it might seem that with all the above simplifications, there is again an issue of arbitrariness, implying that one of the field component is completely independent of the other two [2, 15, 55], but if the wave equations are written in matrix form

$$\begin{bmatrix} k_y^2 & -jk_y\alpha & 0 \\ -jk_y\alpha & -\alpha^2 & 0 \\ 0 & 0 & -\alpha^2 + k_y^2 \end{bmatrix} = k_0^2 \begin{bmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & \epsilon_{yz} \\ 0 & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}, \tag{4.76}$$

which shows that for an anisotropic case, even though the left side of the z component of the wave equation is just in E_{0z}, the right side still has E_{0y}, implying that all the three fields are related. This was not the case when the isotropic behavior was studied, since the wave equations were of the form

$$\begin{bmatrix} k_y^2 & -jk_y\alpha & 0 \\ -jk_y\alpha & -\alpha^2 & 0 \\ 0 & 0 & -\alpha^2 + k_y^2 \end{bmatrix} = k_0^2 \begin{bmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix}, \tag{4.77}$$

which implied that E_{0z} was completely independent of the x and y components [2, 21, 54]. That is why the dispersion relation was found using a TM/TE decomposition. In short, having cross terms in the permittivity tensor ensures that the wave solution for the three components would be tied down and one would not have to resort to using TM/TE

decomposition as done by Wallis et al. [12].

To find the roots for α, we need to find the secular determinant of the matrix formed from eqs. (4.73) and (4.74), i.e.,

$$A(\alpha)D(\alpha) - BC = 0, \qquad (4.78)$$

which gives a fourth order polynomial [12]. So, we have a sum of four solutions for $\vec{E}$, given as

$$E_y = \sum_i F_i D(\alpha_i) e^{-\alpha_i x} e^{i(k_y y - \omega t)}, \qquad (4.79)$$

$$E_z = -C \sum_i F_i e^{-\alpha_i x} e^{i(k_y y - \omega t)}, \qquad (4.80)$$

$$E_x = \sum_i \frac{j k_y \alpha_i F_i D(\alpha_i)}{\kappa^2} e^{-\alpha_i x} e^{j(k_y y - \omega t)}, \qquad (4.81)$$

where the coefficient C has been pulled out of the summation since it is no longer a function of α. The solutions in free space (medium 2) are

$$E_{y2}(x, y; \omega) = E_{1y} e^{-\beta x} e^{j(k_y y - \omega t)}, \qquad (4.82)$$

$$E_{z2}(x, y; \omega) = E_{1z} e^{-\beta x} e^{j(k_y y - \omega t)}, \qquad (4.83)$$

and using the divergence equation $\vec{\nabla} \cdot \vec{E} = 0$, we have E_x as

$$E_{x2}(x, y; \omega) = \frac{j k_y E_{1y}}{\beta} e^{-\beta x} e^{j(k_y y - \omega t)}. \qquad (4.84)$$

Having simplified the field equations inside the substrate and having found all the field values in free space, we can now apply the boundary conditions. Applying the boundary conditions, i.e., tangential electric field on the ground plane is zero and continous at the

interface [24, 52], we get

$$E_{y1}\mid_{x=0} \Rightarrow F_1 D(\alpha_1) + F_2 D(\alpha_2) + F_3 D(\alpha_3) + F_4 D(\alpha_4) \quad = \quad 0, \tag{4.85}$$

$$E_{z1}\mid_{x=0} \Rightarrow -C(F_1 + F_2 + F_3 + F_4) \quad = \quad 0, \tag{4.86}$$

and

$$E_{y1} = E_{y2}\mid_{x=d} \Rightarrow F_1 D(\alpha_1)e^{-\alpha_1 d} + F_2 D(\alpha_2)e^{-\alpha_2 d} + F_3 D(\alpha_3)e^{-\alpha_3 d} + F_4 D(\alpha_4)e^{-\alpha_4 d}$$

$$= E_{1y}e^{-\beta d}, \tag{4.87}$$

$$E_{z1} = E_{z2}\mid_{x=d} \Rightarrow -C(F_1 e^{-\alpha_1 d} + F_2 e^{-\alpha_2 d} + F_3 e^{-\alpha_3 d} + F_4 e^{-\alpha_4 d})$$

$$= E_{1z}e^{-\beta d}. \tag{4.88}$$

Matching the H fields at the interface (the derivation of which has already been done the second chapter), we get

$$H_{y1} = H_{y2}\mid_{x=d} \Rightarrow \frac{1}{j\omega\mu_0} \sum_i -\alpha_i F_i C e^{-\alpha_i d} = \frac{1}{j\omega\mu_0}\beta E_{1z}e^{-\beta d}, \tag{4.89}$$

$$H_{z1} = H_{z2}\mid_{x=d} \Rightarrow \frac{1}{j\omega\mu_0} \sum_i \left(-\alpha_i F_i D(\alpha_i) + \frac{k_y^2}{\kappa^2}\alpha_i F_i D(\alpha_i) \right) e^{-\alpha_i d}$$

$$= \frac{-1}{j\omega\mu_0} \left(\beta E_{1y} + \frac{jk_y}{\beta} jk_y E_{1y} \right) e^{-\beta d}, \tag{4.90}$$

where the common factor, $\frac{1}{j\omega\mu_0}$, can be cancelled. Having matched the fields, we have six equations in six unknowns, viz. $F_1, F_2, F_3, F_4, E_{1y}$, and E_{1z}. In matrix form, the dispersion

relation is written as

$$\begin{bmatrix} D(\alpha_1) & D(\alpha_2) & D(\alpha_3) & D(\alpha_4) & 0 & 0 \\ -C & -C & -C & -C & 0 & 0 \\ D_1 & D_2 & D_3 & D_4 & -e^{-\beta d} & 0 \\ -C_1 & -C_2 & -C_3 & -C_4 & 0 & -e^{-\beta d} \\ M_1 & M_2 & M_3 & M_4 & 0 & -\beta e^{-\beta d} \\ N_1 & N_2 & N_3 & N_4 & \frac{\beta^2 - k_y^2}{\beta} e^{-\beta d} & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ E_{1y} \\ E_{1z} \end{bmatrix} = 0, \qquad (4.91)$$

where

$$\begin{aligned} D_1 &= D(\alpha_1)e^{-\alpha_1 d}, \\ D_2 &= D(\alpha_2)e^{-\alpha_2 d}, \\ D_3 &= D(\alpha_3)e^{-\alpha_3 d}, \\ D_4 &= D(\alpha_4)e^{-\alpha_4 d}. \end{aligned} \qquad (4.92)$$

$$\begin{aligned} C_1 &= Ce^{-\alpha_1 d}, \\ C_2 &= Ce^{-\alpha_2 d}, \\ C_3 &= Ce^{-\alpha_3 d}, \\ C_4 &= Ce^{-\alpha_4 d}. \end{aligned} \qquad (4.93)$$

Also,

$$\begin{aligned} M_1 &= -\alpha_1 Ce^{-\alpha_1 d}, \\ M_2 &= -\alpha_2 Ce^{-\alpha_2 d}, \\ M_3 &= -\alpha_3 Ce^{-\alpha_3 d}, \\ M_4 &= -\alpha_4 Ce^{-\alpha_4 d}, \end{aligned} \qquad (4.94)$$

and

$$N_1 = \alpha_1 D(\alpha_1) \left(-1 + \frac{k_y^2}{\kappa^2} \right) e^{-\alpha_1 d},$$

$$N_2 = \alpha_2 D(\alpha_2) \left(-1 + \frac{k_y^2}{\kappa^2} \right) e^{-\alpha_2 d},$$

$$N_3 = \alpha_3 D(\alpha_3) \left(-1 + \frac{k_y^2}{\kappa^2} \right) e^{-\alpha_3 d},$$

$$N_4 = \alpha_4 D(\alpha_4) \left(-1 + \frac{k_y^2}{\kappa^2} \right) e^{-\alpha_4 d}. \tag{4.95}$$

The matrix, as it stands, is a 6×6 matrix. For a matrix of this order, the secular determinant, i.e.,

$$\begin{vmatrix} A & B & C & D & 0 & 0 \\ E & F & G & H & 0 & 0 \\ I & J & K & L & M & 0 \\ N & O & P & Q & 0 & R \\ S & T & U & V & 0 & W \\ X & Y & Z & a & b & 0 \end{vmatrix} = 0,$$

is given as

$$-TAGMRa + TGMRDX - TGRbDI + YAMWPH - YAGMWQ - BSGRbL$$

$$-FAURbL - FCSMRa - FUMRDX + BXMRUH + BNMWZH - TCERbL$$

$$+TCEMRa + YGMWDN + YAGMRV - YGMRDS - TMRCXH - TARbKH$$

$$+TAMRZH + BEPMWa + TAGRbL + FCSRbL + FCXMRV + BEURbL$$

$$-YMWPDE - BXMWPH + BEZMRV + FPMWDX + FURbDI + TRbKDE$$

$$+FAZMWQ - FCIRbV + FKWbDN + BEKWbQ - BEPWbL + JARbUH$$

$$-JRbUDE - OAGWbL - JRbCSH + JCERbV + JWbPDE - JCEWbQ$$

$$-JAWbPH + FCIWbQ - FAZMRV - JAGRbV + JGRbDS + TRbCIH$$

$$-TMRZDE - YMWCNH - YAMRUH + JWbCNH - BSMRZH + BSRbKH$$

$$+BSGMRa - BIGWbQ + BIGRbV + YMRUDE + YCEMWQ - YCEMRV$$

$$+FAUMRa + FZMRDS - FCXMWQ - FZMWDN + BXGMWQ + JAGWbQ$$

$$-JGWbDN + OAWbKH - BEKRbV - BXGMRV + YMRCSH - BIRbUH$$

$$+BIWbPH - OWbKDE - OWbCIH - OAMWZH - OGMWDX + OAGMWa$$

$$+OCEWbL - OCEMWa + OGWbDI + FAPWbL - FAKWbQ + OMWZDE$$

$$+OMWCXH + BNGWbL - BNWbKH - BNGMWa - BEZMWQ + FAKRbV$$

$$-FKRbDS - FCNWbL - FAPMWa - BEUMRa + FCNMWa - FPWbDI = 0$$

by MATLAB.

Therefore, as mentioned in the introductory chapter, we will focus the analysis to only specific modes which have backward and forward traveling waves with equal amplitudes. This means that since $\alpha_1 = -\alpha_2$ and $\alpha_3 = -\alpha_4$, the number of unknowns inside the substrate, i.e., F_1, F_2, F_3, and F_4, can be reduced to just F_1 and F_3. The above matrix is then reduced to a 4×4 matrix, the determinant of which can be calculated. Also, the number of unknowns has been reduced to four, implying that we have more boundary conditions than unknowns. So, only the tangential electric field boundary conditions would

be used here [12].

The required matrix is, therefore,

$$
\begin{bmatrix}
D(\alpha_1) + D(-\alpha_1) & D(\alpha_3) + D(-\alpha_3) & 0 & 0 \\
-C & -C & 0 & 0 \\
D_{1v} & D_{3v} & -e^{-\beta d} & 0 \\
-C_{1v} & -C_{3v} & 0 & -e^{-\beta d}
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_3 \\
E_{1y} \\
E_{1z}
\end{bmatrix}
= 0, \tag{4.96}
$$

where

$$
\begin{aligned}
D_{1v} &= D(\alpha_1)e^{-\alpha_1 d} + D(-\alpha_1)e^{\alpha_1 d}, \\
D_{3v} &= D(\alpha_3)e^{-\alpha_3 d} + D(-\alpha_3)e^{\alpha_3 d}, \\
C_{1v} &= C(e^{-\alpha_1 d} + e^{\alpha_1 d}), \\
C_{3v} &= C(e^{-\alpha_3 d} + e^{\alpha_3 d}).
\end{aligned} \tag{4.97}
$$

The determinant of eq. (4.96) gives the dispersion relation.

4.2.2   Results for Voigt Geometry with $\vec{B}_0$ Perpendicular to the Interface

To analyze this case, many subcases were studied including effect of damping and effect of different values of the static magnetic field. With damping, the difference can be seen in the values of the dispersion relation in fig. 4.3.

When zoomed in, the inclusion of damping produced a distinct shift from the undamped case as shown in fig. 4.4, which is also present when $B_0 = 5T$. With different values of static magnetic fields, a cutoff appears for $B_0 = .05T$ which does not show for $B_0 = 5T$ [46], indicated by the arrows in fig. 4.5.

Another interesting feature in the plots is that when zoomed in at lower frequencies for any value of B_0, the cyclotron resonances (Ω_{ce} and Ω_{ch}) can be seen as anti-resonances (place where function goes to a maxima, rather than a minima (for surface wave poles)) shown in fig. 4.6, which are damped out when collisions are included. Also, the lower peak
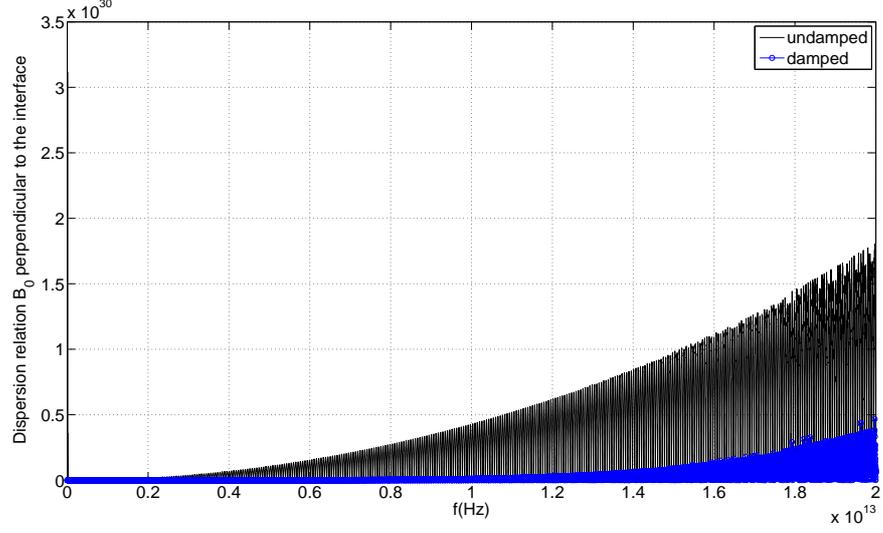
Fig. 4.3: Effect of damping for Voigt geometry with $B_0 = .05T$.

is of Ω_{ce} and the higher of Ω_{ch}.

4.2.3   Dispersion Relation for Faraday Geometry with $\nu << \omega$ and $\vec{B}_0$ Perpendicular to the Interface

Faraday geometry $(\vec{k} \parallel \vec{B}_0)$ [29, 49] enforces only traveling wave components along the $\vec{B}_0$ field. Since $\vec{B}_0$ is perpendicular to the interface, we need to have traveling waves along x direction. For a bonafide surface wave, the wave should attenuate in the direction perpendicular to the interface, i.e., in x direction. Therefore, this case has no usage when studying the surface wave phenomena for $\vec{B}_0$ along x direction. This special case will be utilized while deriving surface wave dispersion relation for $\vec{B}_0$ parallel to the interface, i.e., in y direction.

4.3 $\vec{B}_0$ Parallel to the Interface with $\nu << \omega$

To analyze this case, we first need to simplify the permittivity tensor $\tilde{\epsilon}_p$ for $\theta = 90°$ [12, 35]. Since the tensor is specified by the equation

$$\tilde{\epsilon}_p(\omega) = I - \frac{\omega_{pe}^2}{\omega^2} M_e^{-1} - \frac{\omega_{ph}^2}{\omega} M_h^{-1}, \qquad (4.98)$$
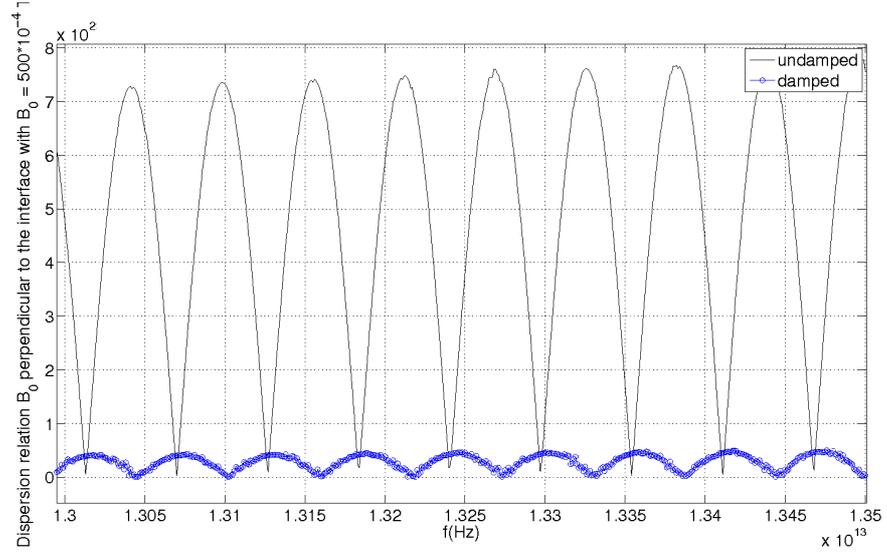
Fig. 4.4: Effect of damping for Voigt geometry with $B_0 = .05T$ zoomed in.

where

$$M_e^{-1} = \frac{1}{\Delta_e} \begin{bmatrix} 1 - X_e^2 cos^2\theta & -X_e^2 cos\theta sin\theta & jX_e sin\theta \\ -X_e^2 cos\theta sin\theta & 1 - X_e^2 sin^2\theta & -jX_e cos\theta \\ -jX_e sin\theta & jX_e cos\theta & 1 \end{bmatrix}, \qquad (4.99)$$

$$M_h^{-1} = \frac{1}{\Delta_h} \begin{bmatrix} 1 - X_h^2 cos^2\theta & -X_h^2 cos\theta sin\theta & -jX_h sin\theta \\ -X_h^2 cos\theta sin\theta & 1 - X_h^2 sin^2\theta & jX_h cos\theta \\ jX_h sin\theta & -jX_h cos\theta & 1 \end{bmatrix}, \qquad (4.100)$$

and

$$\Delta_e = 1 - X_e^2, \qquad (4.101)$$

$$\Delta_h = 1 - X_h^2, \qquad (4.102)$$
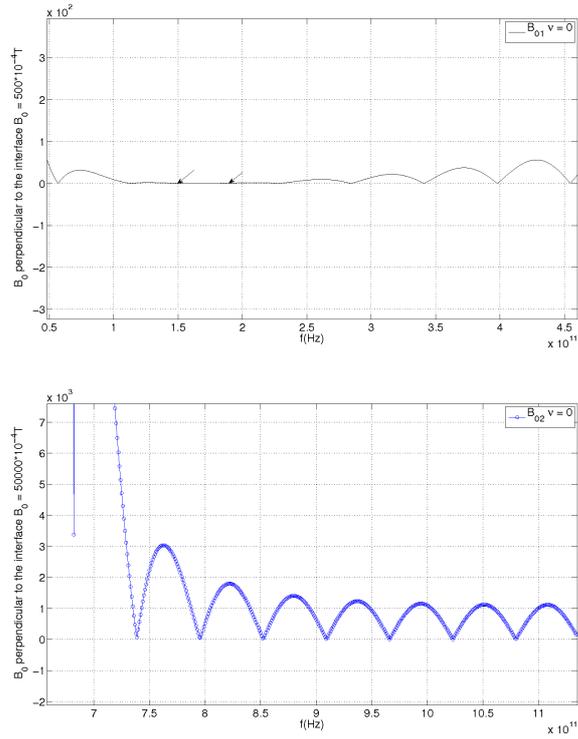
Fig. 4.5: Low frequency cutoff for $B_0 = .5T$.

are the respective determinants of M_e and M_h. $\theta = 90°$ is now substituted in the above matrices to get

$$M_e^{-1} = \frac{1}{\Delta_e} \begin{bmatrix} 1 & 0 & jX_e \\ 0 & 1 - X_e^2 & 0 \\ -jX_e & 0 & 1 \end{bmatrix}, \qquad (4.103)$$

$$M_h^{-1} = \frac{1}{\Delta_h} \begin{bmatrix} 1 & 0 & -jX_h \\ 0 & 1 - X_h^2 & 0 \\ jX_h & 0 & 1 \end{bmatrix}, \qquad (4.104)$$
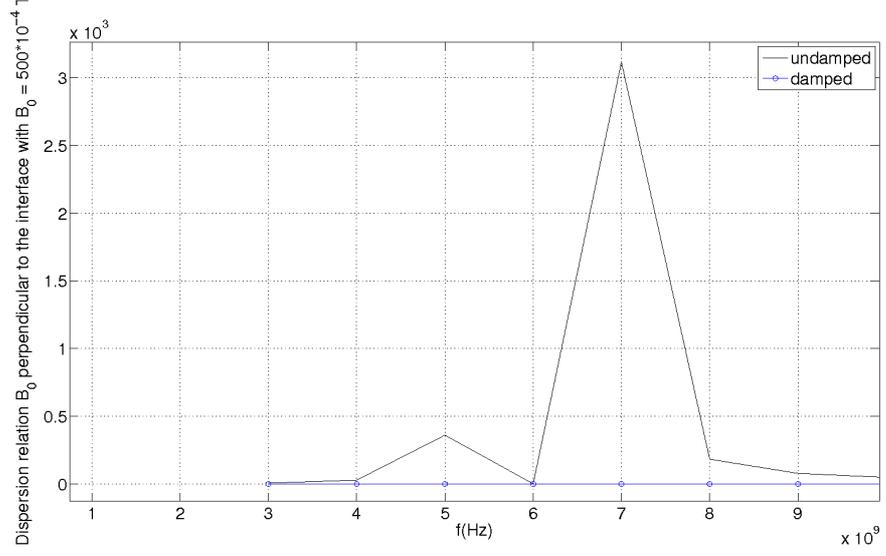
Fig. 4.6: Hole and electron cyclotron resonances for $B_0 = .05T$ zoomed in.

or through further simplification to get

$$M_e^{-1} = \begin{bmatrix} \frac{1}{1-X_e^2} & 0 & \frac{jX_e}{1-X_e^2} \\ 0 & 1 & 0 \\ -\frac{jX_e}{1-X_e^2} & 0 & \frac{1}{1-X_e^2} \end{bmatrix}, \qquad (4.105)$$

$$M_h^{-1} = \begin{bmatrix} \frac{1}{1-X_h^2} & 0 & -\frac{jX_h}{1-X_h^2} \\ 0 & 1 & 0 \\ \frac{jX_h}{1-X_h^2} & 0 & \frac{1}{1-X_h^2} \end{bmatrix}. \qquad (4.106)$$

Then, $\tilde{\epsilon}_p$ can be written as

$$\tilde{\epsilon}_p(\omega) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{\omega_{pe}^2}{\omega^2} \begin{bmatrix} \frac{1}{1-X_e^2} & 0 & \frac{jX_e}{1-X_e^2} \\ 0 & 1 & 0 \\ -\frac{jX_e}{1-X_e^2} & 0 & \frac{1}{1-X_e^2} \end{bmatrix} - \frac{\omega_{ph}^2}{\omega^2} \begin{bmatrix} \frac{1}{1-X_h^2} & 0 & -\frac{jX_h}{1-X_h^2} \\ 0 & 1 & 0 \\ \frac{jX_h}{1-X_h^2} & 0 & \frac{1}{1-X_h^2} \end{bmatrix}, \qquad (4.107)$$

or as

$$\tilde{\epsilon}_p(\omega) = \begin{bmatrix} 1 - \frac{\frac{\omega_{pe}^2}{\omega^2}}{1-X_e^2} - \frac{\frac{\omega_{ph}^2}{\omega^2}}{1-X_h^2} & 0 & -\frac{\frac{jX_e\omega_{pe}^2}{\omega^2}}{1-X_e^2} + \frac{\frac{jX_h\omega_{ph}^2}{\omega^2}}{1-X_h^2} \\ 0 & 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{ph}^2}{\omega^2} & 0 \\ \frac{\frac{jX_e\omega_{pe}^2}{\omega^2}}{1-X_e^2} - \frac{\frac{jX_h\omega_{ph}^2}{\omega^2}}{1-X_h^2} & 0 & 1 - \frac{\frac{\omega_{pe}^2}{\omega^2}}{1-X_e^2} - \frac{\frac{\omega_{ph}^2}{\omega^2}}{1-X_h^2} \end{bmatrix}. \quad (4.108)$$

So, the form of $\tilde{\epsilon}_p$ is

$$\tilde{\epsilon}_p(\omega) = \begin{bmatrix} \epsilon_{xx} & 0 & \epsilon_{xz} \\ 0 & \epsilon_{yy} & 0 \\ \epsilon_{zx} & 0 & \epsilon_{zz} \end{bmatrix}. \quad (4.109)$$

Having found the permittivity tensor for $\theta = 90°$, the analysis can now be narrowed down to Voigt ($\vec{k} \perp \vec{B}_0$) [12] and Faraday ($\vec{k} \parallel \vec{B}_0$) [29,49] geometries as done in the previous section. Dispersion relation for these simplified cases with $\nu << \omega$ has been done here, since the form of the permittivity tensor would still be the same when collisions are not neglected. Although, there would be some extra terms in the tensor elements, the tensor as a whole would still have the same elements going to zero for different configurations of $\vec{B}_0$. This implies that the same dispersion relations will hold for the cases where $\nu \simeq \omega$ and it would not be necessary to derive them again.

4.3.1 Dispersion Relation for Voigt Geometry with $\nu << \omega$ and $\vec{B}_0$ Parallel to the Interface

For this geometry, $\vec{k} \perp \vec{B}_0$, with $\vec{B}_0 = \hat{y}B_0$. The traveling wave numbers are k_y and k_z. Out of the two, we can only chose that wave number that is perpendicular to the steady magnetic field along $\hat{y}$ direction. We cannot have traveling waves along x direction, since that would not correspond to a surface wave, we only have k_z which is perpendicular to $\hat{y}$. The other wave number, k_y, is along B_0 and so cannot be considered for this case.

Inside the substrate, we have

$$A(\alpha) = (-\alpha^2 + k_z^2 - k_0^2\epsilon_{yy})\kappa^2, \tag{4.110}$$

$$B = 0, \tag{4.111}$$

$$C = 0, \tag{4.112}$$

$$D(\alpha) = (-\alpha^2 - k_0^2\epsilon_{zz})\kappa^2 - (-k_z^2\alpha^2 + jk_z\alpha k_0^2\epsilon_{xz} + jk_z\alpha k_0^2\epsilon_{zx} \tag{4.113}$$

$$+k_0^4\epsilon_{zx}\epsilon_{xz}),$$

implying that the pair of linear simultaneous equations, viz.

$$A(\alpha)E_{0y} + BE_{0z} = 0, \tag{4.114}$$

$$CE_{0y} + D(\alpha)E_{0z} = 0, \tag{4.115}$$

will be reduced to

$$A(\alpha)E_{0y} = 0, \tag{4.116}$$

$$D(\alpha)E_{0z} = 0. \tag{4.117}$$

Thus, we find that this configuration reduces to a degenerate case. This becomes clear when the wave equation $\vec{\nabla} \times \vec{\nabla} \times \vec{E}_1 = k_0^2\tilde{\epsilon}_p(\omega) \cdot \vec{E}_1$ is written. The equation is

$$\hat{x} \Rightarrow -\alpha(-\alpha E_{0x} + jk_z E_{0z}) - (\alpha^2 - k_z^2)E_{0x} = k_0^2(\epsilon_{xx}E_{0x} + \epsilon_{xz}E_{0z}), \tag{4.118}$$

$$\hat{y} \Rightarrow -(\alpha^2 - k_z^2)E_{0y} = k_0^2\epsilon_{yy}E_{0y}, \tag{4.119}$$

$$\hat{z} \Rightarrow jk_z(-\alpha E_{0x} + jk_z E_{0z}) - (\alpha^2 - k_z^2)E_{0z} = k_0^2(\epsilon_{zx}E_{0x} + \epsilon_{zz}E_{0z}). \tag{4.120}$$

It can be seen that E_{y1} is completely independent of the other two electric fields. The mode turns out to be a TM_z mode if E_{x1} and E_{z1} are solved for and is a TE_z mode if E_{y1} is solved for. In this analysis, therefore, the TM_z mode will be solved, i.e., the fields $E_x, E_z,$

and H_y [35, 49, 54]. Simplifying the x and z components of the wave equation, we get

$$(k_z^2 - k_0^2 \epsilon_{xx}) E_{0x} + (-jk_z \alpha - k_0^2 \epsilon_{xz}) E_{0z} = 0, \tag{4.121}$$

$$(-jk_z \alpha - k_0^2 \epsilon_{zx}) E_{0x} + (-\alpha^2 - k_0^2 \epsilon_{zz}) E_{0z} = 0. \tag{4.122}$$

Equations (4.121) and (4.122) imply that we have two linear equations

$$AE_{0x} + B(\alpha) E_{0z} = 0, \tag{4.123}$$

$$C(\alpha) E_{0x} + D(\alpha) E_{0z} = 0, \tag{4.124}$$

where

$$\begin{aligned} A &= k_z^2 - k_0^2 \epsilon_{xx}, \\ B(\alpha) &= -jk_z \alpha - k_0^2 \epsilon_{xz}, \\ C(\alpha) &= -jk_z \alpha - k_0^2 \epsilon_{zx}, \\ D(\alpha) &= -\alpha^2 - k_0^2 \epsilon_{zz}. \end{aligned} \tag{4.125}$$

Equating the determinant of the matrix

$$\begin{bmatrix} A & B(\alpha) \\ C(\alpha) & D(\alpha) \end{bmatrix}, \tag{4.126}$$

equal to zero, i.e.,

$$\begin{vmatrix} A & B(\alpha) \\ C(\alpha) & D(\alpha) \end{vmatrix} = 0, \tag{4.127}$$

gives a polynomial equation in α. Since A is independent of α and B, C, D have powers of 1, 1, and 2, respectively, the power of the equation would be two. Some interesting facts can be derived from this. Although the medium is anisotropic, the orientation of $\vec{B}_0$ has had a vast impact on the field solutions. One would have expected the number of roots (α) to be four for a general anisotropic medium, but in this case, the medium does not behave

as expected [12].

The solution can be written as either

$$E_{0x} = \sum_i F_i B(\alpha_i), \tag{4.128}$$

$$E_{0z} = -A \sum_i F_i, \tag{4.129}$$

or

$$E_{0x} = \sum_i F_i D(\alpha_i), \tag{4.130}$$

$$E_{0z} = -\sum_i F_i C(\alpha_i). \tag{4.131}$$

The full solution in medium 1 (semiconductor) can be written as

$$E_{x1}(x, z; \omega) = \sum_i F_i D(\alpha_i) e^{-\alpha_i x} e^{j(k_z z - \omega t)}, \tag{4.132}$$

$$E_{z1}(x, z; \omega) = -\sum_i F_i C(\alpha_i) e^{-\alpha_i x} e^{j(k_z z - \omega t)}, \tag{4.133}$$

where the second solution, i.e., eqs. (4.130) and (4.131), of the pair of algebraic equations has been used. The unknowns are F_1, F_2, and E_{1z} for this TM_z mode [15, 56]. To get a unique dispersion relation, the tangential fields at $x = 0$ and $x = d$ are to be matched. We also need H_{y1}, which can be found from the y component of Maxwell's curl equation $\vec{\nabla} \times \vec{E}_1 = j\omega\mu_0 \vec{H}_1$ to get

$$-(-\alpha E_{0z} - jk_z E_{0x}) = j\omega\mu_0 H_{0y}, \tag{4.134}$$

$$\Rightarrow H_{0y} = \frac{\alpha E_{0z} + jk_z E_{0x}}{j\omega\mu_0}. \tag{4.135}$$

The full solution for H_{y1} is then given by

$$H_{y1}(x, z; \omega) = \frac{1}{j\omega\mu_0} \sum_i F_i \left(-\alpha_i C(\alpha_i) + jk_z D(\alpha_i) \right) e^{-\alpha_i x} e^{j(k_z z - \omega t)}. \tag{4.136}$$

In medium 2, the required field equations are

$$E_{z2}(x, z; \omega) = E_{1z}e^{-\beta x}e^{j(k_z z - \omega t)}, \tag{4.137}$$

$$E_{x2}(x, z; \omega) = \frac{jk_z}{\beta}E_{1z}e^{-\beta x}e^{j(k_z z - \omega t)}, \tag{4.138}$$

$$H_{y2}(x, z; \omega) = \frac{\beta^2 - k_z^2}{j\beta\omega\mu_0}E_{1z}e^{-\beta x}e^{j(k_z z - \omega t)}, \tag{4.139}$$

where E_{x2} is related to E_{z2} through $\vec{\nabla} \cdot \vec{E}_2 = 0$ (nonexistence of free charges in medium 2) and H_{y2} through the y component of $\vec{\nabla} \times \vec{E}_2 = j\omega\mu_0\vec{H}_2$.

Having found all the required fields in both the mediums, they can be matched across the interface and at $x = 0$. Using $\vec{E}_{tan}|_{x=0} = 0$ gives

$$F_1 C(\alpha_1) + F_2 C(\alpha_2) = 0, \tag{4.140}$$

and matching the tangential fields at $x = d$ gives

$$-F_1 C(\alpha_1)e^{-\alpha_1 d} - F_2 C(\alpha_2)e^{-\alpha_2 d} = E_{1z}e^{-\beta d}, \tag{4.141}$$

$$F_1(-\alpha_1 C(\alpha_1) + jk_z D(\alpha_1))e^{-\alpha_1 d} + F_2(-\alpha_2 C(\alpha_2) + jk_z D(\alpha_2))e^{-\alpha_2 d}$$
$$= \frac{\beta^2 - k_z^2}{\beta}E_{1z}e^{-\beta d}. \tag{4.142}$$

Finally, the above equations can be written in matrix form

$$\begin{bmatrix} C(\alpha_1) & C(\alpha_2) & 0 \\ C(\alpha_1)e^{-\alpha_1 d} & C(\alpha_2)e^{-\alpha_2 d} & e^{-\beta d} \\ L_1 & L_2 & -\frac{\beta^2 - k_z^2}{\beta}e^{-\beta d} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ E_{1z} \end{bmatrix} = 0, \tag{4.143}$$

where

$$L_1 = (-\alpha_1 C(\alpha_1) + jk_z D(\alpha_1))e^{-\alpha_1 d}, \tag{4.144}$$

$$L_2 = (-\alpha_2 C(\alpha_2) + jk_z D(\alpha_2))e^{-\alpha_2 d}. \tag{4.145}$$

Equating the secular determinant of eq. (4.143) to zero will result in the required dispersion relation.

4.3.2 Dispersion Relation for Faraday Geometry with $\nu << \omega$ and $\vec{B}_0$ Parallel to the Interface

For this geometry, $\vec{k} \parallel \vec{B}_0$, with $\vec{B}_0 = \hat{y}B_0$ [35, 49]. This geometry implies that the propagating wave number parallel to B_0 is k_y. So, the equations derived for a general dispersion relation can be simplified using the following information

$$
\begin{aligned}
k_z &= 0, \\
\epsilon_{xy} &= 0, \\
\epsilon_{yx} &= 0, \\
\epsilon_{yz} &= 0, \\
\epsilon_{zy} &= 0.
\end{aligned}
\tag{4.146}
$$

Inside the semiconductor, solving the three components of the wave equation, we have the following coefficients of E_{0y} and E_{0z}:

$$
\begin{aligned}
A(\alpha) = (-\alpha^2 + k_z^2 - k_0^2\epsilon_{yy})\kappa^2 - (-k_y^2\alpha^2 + jk_y\alpha k_0^2\epsilon_{xy} + jk_y\alpha k_0^2\epsilon_{yx} \\
+ k_0^4\epsilon_{yx}\epsilon_{xy}),
\end{aligned}
\tag{4.147}
$$

$$
\begin{aligned}
B(\alpha) = (-k_z k_y - k_0^2\epsilon_{yz})\kappa^2 - (-k_y k_z\alpha^2 + jk_y\alpha k_0^2\epsilon_{xz} + jk_z\alpha k_0^2\epsilon_{yx} \\
+ k_0^4\epsilon_{yx}\epsilon_{xz}),
\end{aligned}
\tag{4.148}
$$

$$
\begin{aligned}
C(\alpha) = (-k_y k_z - k_0^2\epsilon_{zy})\kappa^2 - (-k_y k_z\alpha^2 + jk_z\alpha k_0^2\epsilon_{xy} + jk_y\alpha k_0^2\epsilon_{zx} \\
+ k_0^4\epsilon_{zx}\epsilon_{xy}),
\end{aligned}
\tag{4.149}
$$

$$
\begin{aligned}
D(\alpha) = (-\alpha^2 + k_y^2 - k_0^2\epsilon_{zz})\kappa^2 - (-k_z^2\alpha^2 + jk_z\alpha k_0^2\epsilon_{xz} + jk_z\alpha k_0^2\epsilon_{zx} \\
+ k_0^4\epsilon_{zx}\epsilon_{xz}),
\end{aligned}
\tag{4.150}
$$

which have already been derived in the second chapter.

Using eq. (4.146), A, B, C, and D can be reduced to

$$A(\alpha) = (-\alpha^2 - k_0^2 \epsilon_{yy})\kappa^2 - (-k_y^2 \alpha^2), \tag{4.151}$$

$$B(\alpha) = -(jk_y \alpha k_0^2 \epsilon_{xz}), \tag{4.152}$$

$$C(\alpha) = -(jk_y \alpha k_0^2 \epsilon_{zx}), \tag{4.153}$$

$$D(\alpha) = (-\alpha^2 + k_y^2 - k_0^2 \epsilon_{zz})\kappa^2 - (k_0^4 \epsilon_{zx} \epsilon_{xz}), \tag{4.154}$$

where $\kappa = \sqrt{(k_y^2 - k_0^2 \epsilon_{xx})}$. The first observation is that for this simplifying case, all the four coefficients are functions of α. E_{0y} and E_{0z} are realted through the linear equations

$$A(\alpha)E_{0y} + B(\alpha)E_{0z} = 0, \tag{4.155}$$

$$C(\alpha)E_{0y} + D(\alpha)E_{0z} = 0, \tag{4.156}$$

and the remaining field component E_{0x} is related to E_{0y} and E_{0z} through the equation

$$E_{0x} = \frac{jk_y \alpha E_{0y} + k_0^2 \epsilon_{xz} E_{0z}}{\kappa^2}. \tag{4.157}$$

To check if the field quantities are dependent or independent of each other, the wave equations are written in matrix form to get

$$\begin{bmatrix} k_y^2 & -jk_y \alpha & 0 \\ -jk_y \alpha & -\alpha^2 & 0 \\ 0 & 0 & -\alpha^2 + k_y^2 \end{bmatrix} = k_0^2 \begin{bmatrix} \epsilon_{xx} & 0 & \epsilon_{xz} \\ 0 & \epsilon_{yy} & 0 \\ \epsilon_{zx} & 0 & \epsilon_{zz} \end{bmatrix}, \tag{4.158}$$

which shows that even though the left side of the z component of the wave equation is just in E_{0z}, the right side still has E_{0x}, implying that all the three fields are related, which is

different from the isotropic case [12], which had the form

$$
\begin{bmatrix}
k_y^2 & -jk_y\alpha & 0 \\
-jk_y\alpha & -\alpha^2 & 0 \\
0 & 0 & -\alpha^2 + k_y^2
\end{bmatrix}
= k_0^2
\begin{bmatrix}
\epsilon_{xx} & 0 & 0 \\
0 & \epsilon_{yy} & 0 \\
0 & 0 & \epsilon_{zz}
\end{bmatrix},
\tag{4.159}
$$

which implied that E_{0z} was completely independent of the x and y components.

Having ensured that we are not dealing with any sort of arbitrariness, the roots for α can be found by equating the secular determinant of the matrix formed from eqs. (4.155) and (4.156) to zero, i.e.,

$$
A(\alpha)D(\alpha) - B(\alpha)C(\alpha) = 0.
\tag{4.160}
$$

Equation (4.160) gives a fourth order polynomial in α [12, 29]. So, we have a sum of four solutions for $\vec{E}$, given as

$$
E_{y1} = \sum_i F_i D(\alpha_i) e^{-\alpha_i x} e^{i(k_y y - \omega t)},
\tag{4.161}
$$

$$
E_{z1} = -\sum_i F_i C(\alpha_i) e^{-\alpha_i x} e^{i(k_y y - \omega t)},
\tag{4.162}
$$

$$
E_{x1} = \sum_i \frac{(jk_y\alpha_i)F_i D(\alpha_i) - k_0^2\epsilon_{xz}F_i C(\alpha_i)}{\kappa^2} e^{-\alpha_i x} e^{j(k_y y - \omega t)}.
\tag{4.163}
$$

On the free space we have the field solutions as

$$
E_{y2}(x, y; \omega) = E_{1y} e^{-\beta x} e^{j(k_y y - \omega t)},
\tag{4.164}
$$

$$
E_{z2}(x, y; \omega) = E_{1z} e^{-\beta x} e^{j(k_y y - \omega t)},
\tag{4.165}
$$

and again using the divergence equation $\vec{\nabla} \cdot \vec{E} = 0$, we have E_x as

$$
E_{x2}(x, y; \omega) = \frac{jk_y E_{1y}}{\beta} e^{-\beta x} e^{j(k_y y - \omega t)}.
\tag{4.166}
$$

After simplification, having found all the field equations inside the substrate and in free space, we can now apply the boundary conditions to the fields at the ground plane and

at the interface. Applying the tangential electric field boundary condition on the ground plane and at the interface [52], we get

$$E_{y1}\mid_{x=0} \Rightarrow F_1 D(\alpha_1) + F_2 D(\alpha_2) + F_3 D(\alpha_3) + F_4 D(\alpha_4) \quad = \quad 0, \qquad (4.167)$$

$$E_{z1}\mid_{x=0} \Rightarrow -(F_1 C(\alpha_1) + F_2 C(\alpha_2) + F_3 C(\alpha_3) + F_4 C(\alpha_4)) \quad = \quad 0, \qquad (4.168)$$

and

$$E_{y1} = E_{y2}\mid_{x=d} \Rightarrow F_1 D(\alpha_1)e^{-\alpha_1 d} + F_2 D(\alpha_2)e^{-\alpha_2 d} + F_3 D(\alpha_3)e^{-\alpha_3 d}$$
$$+ F_4 D(\alpha_4)e^{-\alpha_4 d} = E_{1y}e^{-\beta d}, \qquad (4.169)$$

$$E_{z1} = E_{z2}\mid_{x=d} \Rightarrow -(F_1 C(\alpha_1)e^{-\alpha_1 d} + F_2 C(\alpha_2)e^{-\alpha_2 d} + F_3 C(\alpha_3)e^{-\alpha_3 d}$$
$$+ F_4 C(\alpha_4)e^{-\alpha_4 d}) = E_{1z}e^{-\beta d}. \qquad (4.170)$$

Matching of $\vec{H}$ fields at the interface (the derivation of which has already been done in the second chapter) gives the required equations for the number of unknowns. These are

$$H_{y1} = H_{y2}\mid_{x=d} \Rightarrow \frac{1}{j\omega\mu_0}\sum_i[-\alpha_i F_i C(\alpha_i)]e^{-\alpha_i d} = \frac{1}{j\omega\mu_0}[\beta E_{1z}]e^{-\beta d}, \qquad (4.171)$$

$$H_{z1} = H_{z2}\mid_{x=d} \Rightarrow \frac{1}{j\omega\mu_0}\sum_i[-\alpha_i F_i D(\alpha_i) - \frac{jk_y}{\kappa^2}[(jk_y\alpha_i)F_i D(\alpha_i)$$

$$-(k_0^2\epsilon_{xz})F_i C(\alpha_i)]]e^{-\alpha_i d} = \frac{-1}{j\omega\mu_0}[\beta E_{1y} + \frac{jk_y}{\beta}(jk_y E_{1y})]e^{-\beta d}, \qquad (4.172)$$

where the common factor, $\frac{1}{j\omega\mu_0}$, can be cancelled. Having matched the fields, we have six equations in six unknowns, viz. $F_1, F_2, F_3, F_4, E_{1y}, E_{1z}$. In matrix form, the dispersion

relation is written as

$$
\begin{bmatrix}
D(\alpha_1) & D(\alpha_2) & D(\alpha_3) & D(\alpha_4) & 0 & 0 \\
-C(\alpha_1) & -C(\alpha_2) & -C(\alpha_3) & -C(\alpha_4) & 0 & 0 \\
D_{1f} & D_{2f} & D_{3f} & D_{4f} & -e^{-\beta d} & 0 \\
-C_{1f} & -C_{2f} & -C_{3f} & -C_{4f} & 0 & -e^{-\beta d} \\
M_1 & M_2 & M_3 & M_4 & 0 & -\beta e^{-\beta d} \\
N_1 & N_2 & N_3 & N_4 & \frac{\beta^2 - k_y^2}{\beta} e^{-\beta d} & 0
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
F_4 \\
E_{1y} \\
E_{1z}
\end{bmatrix}
= 0, \qquad (4.173)
$$

where

$$
\begin{aligned}
D_{1f} &= D(\alpha_1)e^{-\alpha_1 d}, \\
D_{2f} &= D(\alpha_2)e^{-\alpha_2 d}, \\
D_{3f} &= D(\alpha_3)e^{-\alpha_3 d}, \\
D_{4f} &= D(\alpha_4)e^{-\alpha_4 d}, \qquad (4.174)
\end{aligned}
$$

$$
\begin{aligned}
C_{1f} &= C(\alpha_1)e^{-\alpha_1 d}, \\
C_{2f} &= C(\alpha_2)e^{-\alpha_2 d}, \\
C_{3f} &= C(\alpha_3)e^{-\alpha_3 d}, \\
C_{4f} &= C(\alpha_4)e^{-\alpha_4 d}. \qquad (4.175)
\end{aligned}
$$

Also,

$$
\begin{aligned}
M_1 &= -\alpha_1 C(\alpha_1)e^{-\alpha_1 d}, \\
M_2 &= -\alpha_2 C(\alpha_2)e^{-\alpha_2 d}, \\
M_3 &= -\alpha_3 C(\alpha_3)e^{-\alpha_3 d}, \\
M_4 &= -\alpha_4 C(\alpha_4)e^{-\alpha_4 d}, \qquad (4.176)
\end{aligned}
$$

and

$$N_1 = \left(-\alpha_1 D(\alpha_1) - \frac{jk_y}{\kappa^2}(jk_y\alpha_1 D(\alpha_1) - k_0^2\epsilon_{xz}C(\alpha_1))\right)e^{-\alpha_1 d},$$

$$N_2 = \left(-\alpha_2 D(\alpha_2) - \frac{jk_y}{\kappa^2}(jk_y\alpha_2 D(\alpha_2) - k_0^2\epsilon_{xz}C(\alpha_2))\right)e^{-\alpha_2 d},$$

$$N_3 = \left(-\alpha_3 D(\alpha_3) - \frac{jk_y}{\kappa^2}(jk_y\alpha_3 D(\alpha_3) - k_0^2\epsilon_{xz}C(\alpha_3))\right)e^{-\alpha_3 d},$$

$$N_4 = \left(-\alpha_4 D(\alpha_4) - \frac{jk_y}{\kappa^2}(jk_y\alpha_4 D(\alpha_4) - k_0^2\epsilon_{xz}C(\alpha_4))\right)e^{-\alpha_4 d}. \qquad (4.177)$$

As before, we have to reduce the number of unknowns using the simplification that $\alpha_1 = -\alpha_2$ and $\alpha_3 = -\alpha_4$.

The unknowns are reduced to F_1, F_3, E_{1y}, and E_{1z}. Therefore, the four required relations are

$$F_1(D(\alpha_1) + D(-\alpha_1)) + F_3(D(\alpha_3) + D(-\alpha_3)) = 0, \qquad (4.178)$$

$$F_1(D(\alpha_1)e^{-\alpha_1 d} + D(-\alpha_1)e^{\alpha_1 d}) + F_3(D(\alpha_3)e^{-\alpha_3 d} + D(-\alpha_3)e^{\alpha_3 d}) = E_{1y}e^{-\beta d},$$

$$(4.179)$$

$$-[F_1(C(\alpha_1)e^{-\alpha_1 d} + C(-\alpha_1)e^{\alpha_1 d}) + F_3(C(\alpha_3)e^{-\alpha_3 d} + C(-\alpha_3)e^{\alpha_3 d})] = E_{1z}e^{-\beta d},$$

$$(4.180)$$

and

$$F_1[(-\alpha_1 D(\alpha_1) - \frac{jk_y}{\kappa^2}[jk_y\alpha_1 D(\alpha_1) - k_0^2\epsilon_{xz}C(\alpha_1)])e^{-\alpha_1 d} + (\alpha_1 D(-\alpha_1)$$

$$-\frac{jk_y}{\kappa^2}[-jk_y\alpha_1 D(-\alpha_1) - k_0^2\epsilon_{xz}C(-\alpha_1)])e^{\alpha_1 d}]$$

$$+F_3[(-\alpha_3 D(\alpha_3) - \frac{jk_y}{\kappa^2}[jk_y\alpha_3 D(\alpha_3) - k_0^2\epsilon_{xz}C(\alpha_3)])e^{-\alpha_3 d} + (\alpha_3 D(-\alpha_3)$$

$$-\frac{jk_y}{\kappa^2}[-jk_y\alpha_3 D(-\alpha_3) - k_0^2\epsilon_{xz}C(-\alpha_3)])e^{\alpha_3 d}] = -[\beta E_{1y} + \frac{jk_y}{\beta}(jk_y E_{1y})]e^{-\beta d},$$

$$(4.181)$$

which can be written in matrix form as

$$
\begin{bmatrix}
D(\alpha_1) + D(-\alpha_1) & D(\alpha_3) + D(-\alpha_3) & 0 & 0 \\
M_1 & M_3 & -e^{-\beta d} & 0 \\
N_1 & N_3 & 0 & e^{-\beta d} \\
P_1 & P_3 & \frac{\beta^2 - k_y^2}{\beta} e^{-\beta d} & 0
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_3 \\
E_{1y} \\
E_{1z}
\end{bmatrix}
= 0,
\qquad (4.182)
$$

where

$$
M_1 = D(\alpha_1)e^{-\alpha_1 d} + D(-\alpha_1)e^{\alpha_1 d}, \qquad (4.183)
$$

$$
M_3 = D(\alpha_3)e^{-\alpha_3 d} + D(-\alpha_3)e^{\alpha_3 d}, \qquad (4.184)
$$

$$
N_1 = C(\alpha_1)e^{-\alpha_1 d} + C(-\alpha_1)e^{\alpha_1 d}, \qquad (4.185)
$$

$$
N_3 = C(\alpha_3)e^{-\alpha_3 d} + C(-\alpha_3)e^{\alpha_3 d}, \qquad (4.186)
$$

and

$$
P_1 = (-\alpha_1 D(\alpha_1) - \frac{jk_y}{\kappa^2}[jk_y \alpha_1 D(\alpha_1) - k_0^2 \epsilon_{xz} C(\alpha_1)])e^{-\alpha_1 d}
$$
$$
+(\alpha_1 D(-\alpha_1) - \frac{jk_y}{\kappa^2}[-jk_y \alpha_1 D(-\alpha_1) - k_0^2 \epsilon_{xz} C(-\alpha_1)])e^{\alpha_1 d}, \qquad (4.187)
$$
$$
P_3 = (-\alpha_3 D(\alpha_3) - \frac{jk_y}{\kappa^2}[jk_y \alpha_3 D(\alpha_3) - k_0^2 \epsilon_{xz} C(\alpha_3)])e^{-\alpha_3 d}
$$
$$
+(\alpha_3 D(-\alpha_3) - \frac{jk_y}{\kappa^2}[-jk_y \alpha_3 D(-\alpha_3) - k_0^2 \epsilon_{xz} C(-\alpha_3)])e^{\alpha_3 d}. \qquad (4.188)
$$

The following sections will have some more information about the medium through the permittivity tensor $\tilde{\epsilon}_p$, but their forms will still be the same with some minor changes to the tensor elements. In the sections below, only the final dispersion curves will be given. The plots for dispersion relation are listed in the section when damping is considered for the same orientation of magnetic field.

4.4 $\vec{B}_0$ Perpendicular to the Interface

In this section, emphasis will be put on the derivation of $\tilde{\epsilon}_p$, with $\vec{B}_0 = \hat{x}B_0$, i.e., $\theta = 0°$,

without neglecting the damping terms ν_{eh} and ν_{he}. The current density equation can be written as

$$\vec{J} = -N_0 q \left(I + \frac{j\nu_{eh}}{\omega} M_e^{-1} + \frac{j\nu_{he}}{\omega} M_h^{-1} \right)^{-1} \left(-\frac{jq}{\omega m_e} M_e^{-1} - \frac{jq}{\omega m_h} M_h^{-1} \right) \cdot \vec{E}. \qquad (4.189)$$

With $\theta = 0°$, M_e and M_h become

$$M_e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{j\Omega_{ce}}{\omega} \\ 0 & \frac{-j\Omega_{ce}}{\omega} & 1 \end{bmatrix}, \qquad (4.190)$$

$$\Rightarrow M_e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & jX_e \\ 0 & -jX_e & 1 \end{bmatrix}, \qquad (4.191)$$

$$M_h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{-j\Omega_{ch}}{\omega} \\ 0 & \frac{j\Omega_{ch}}{\omega} & 1 \end{bmatrix}, \qquad (4.192)$$

$$\Rightarrow M_h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -jX_h \\ 0 & jX_h & 1 \end{bmatrix}, \qquad (4.193)$$

where X_e and X_h stand for the ratio $\frac{\Omega_{ce}}{\omega}$ and $\frac{\Omega_{ch}}{\omega}$, respectively, as before. The inverse of M_e and M_h are, for the present orientation of $\vec{B}_0$, leads to

$$M_e^{-1} = \frac{1}{\Delta_e} \begin{bmatrix} 1 - X_e^2 & 0 & 0 \\ 0 & 1 & -jX_e \\ 0 & jX_e & 1 \end{bmatrix}, \qquad (4.194)$$

and

$$M_h^{-1} \;=\; \frac{1}{\Delta_h} \begin{bmatrix} 1 - X_h^2 & 0 & 0 \\ 0 & 1 & jX_h \\ 0 & -jX_h & 1 \end{bmatrix}. \tag{4.195}$$

First, we need to find the matrix $[I + \frac{j\nu_{eh}}{\omega} M_e^{-1} + \frac{j\nu_{he}}{\omega} M_h^{-1}]^{-1}$ or $[I + j\Gamma_e M_e^{-1} + j\Gamma_h M_h^{-1}]^{-1}$. Having already found the inverses, we just need to add them and take the inverse of the matrix [64, 65]. We, therefore, get

$$I + j\Gamma_e M_e^{-1} + j\Gamma_h M_h^{-1} \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} j\Gamma_e & 0 & 0 \\ 0 & \frac{j\Gamma_e}{1-X_e^2} & \frac{X_e\Gamma_e}{1-X_e^2} \\ 0 & \frac{-X_e\Gamma_e}{1-X_e^2} & \frac{j\Gamma_e}{1-X_e^2} \end{bmatrix}$$

$$+ \begin{bmatrix} j\Gamma_h & 0 & 0 \\ 0 & \frac{j\Gamma_h}{1-X_h^2} & \frac{-X_h\Gamma_h}{1-X_h^2} \\ 0 & \frac{X_h\Gamma_h}{1-X_h^2} & \frac{j\Gamma_h}{1-X_h^2} \end{bmatrix}. \tag{4.196}$$

Further simplification yields

$$I + j\Gamma_e M_e^{-1} + j\Gamma_h M_h^{-1} = \begin{bmatrix} 1 + j(\Gamma_e + \Gamma_h) & 0 & 0 \\ 0 & 1 + \frac{j\Gamma_e}{1-X_e^2} + \frac{j\Gamma_h}{1-X_h^2} & \frac{X_e\Gamma_e}{1-X_e^2} - \frac{X_h\Gamma_h}{1-X_h^2} \\ 0 & \frac{X_h\Gamma_h}{1-X_h^2} - \frac{X_e\Gamma_e}{1-X_e^2} & 1 + \frac{j\Gamma_e}{1-X_e^2} + \frac{j\Gamma_h}{1-X_h^2} \end{bmatrix}. \tag{4.197}$$

Let $Y_1 = 1 + j(\Gamma_e + \Gamma_h)$, $Y_2 = 1 + \frac{j\Gamma_e}{1-X_e^2} + \frac{j\Gamma_h}{1-X_h^2}$, $Y_3 = \frac{X_e\Gamma_e}{1-X_e^2} - \frac{X_h\Gamma_h}{1-X_h^2}$, and $A = I + j\Gamma_e M_e^{-1} + j\Gamma_h M_h^{-1}$. Then

$$A = \begin{bmatrix} Y_1 & 0 & 0 \\ 0 & Y_2 & Y_3 \\ 0 & -Y_3 & Y_2 \end{bmatrix}, \tag{4.198}$$

where

$$\Delta_A = Y_1(Y_2^2 + Y_3^2), \tag{4.199}$$

where Δ_A is the determinant of A. We need the inverse of A, which is given as (after multiplying with Δ_A) [65]

$$A^{-1} = \begin{bmatrix} \frac{1}{Y_1} & 0 & 0 \\ 0 & \frac{Y_2}{Y_2^2+Y_3^2} & \frac{-Y_3}{Y_2^2+Y_3^2} \\ 0 & \frac{Y_3}{Y_2^2+Y_3^2} & Y_2Y_2^2 + Y_3^2 \end{bmatrix}. \tag{4.200}$$

The current density equation can be simplified as

$$\vec{J} = (\frac{jN_0q^2}{\omega m_e}A^{-1}M_e^{-1} + \frac{jN_0q^2}{\omega m_h}A^{-1}M_h^{-1}) \cdot \vec{E}, \tag{4.201}$$

and the Maxwell's equation $\vec{\nabla} \times \vec{H}$ becomes

$$\vec{\nabla} \times \vec{H} = (\frac{jN_0q^2}{\omega m_e}A^{-1}M_e^{-1} + \frac{jN_0q^2}{\omega m_h}A^{-1}M_h^{-1} - j\omega\epsilon_0 I) \cdot \vec{E}, \tag{4.202}$$

$$\Rightarrow \vec{\nabla} \times \vec{H} = -j\omega\epsilon_0(I - \frac{\omega_{pe}^2}{\omega^2}A^{-1}M_e^{-1} - \frac{\omega_{ph}^2}{\omega^2}A^{-1}M_h^{-1}) \cdot \vec{E}. \tag{4.203}$$

$\tilde{\epsilon}_p$ can therefore be formed as

$$\tilde{\epsilon}_p(\omega) = I - \frac{\omega_{pe}^2}{\omega^2}A^{-1}M_e^{-1} - \frac{\omega_{ph}^2}{\omega^2}A^{-1}M_h^{-1}, \tag{4.204}$$

where $A^{-1}M_e^{-1}$ and $A^{-1}M_h^{-1}$ are given by

$$A^{-1}M_e^{-1} = \begin{bmatrix} \frac{1}{Y_1} & 0 & 0 \\ 0 & \frac{Y_2-jX_eY_3}{(Y_2^2+Y_3^2)(1-X_e^2)} & \frac{-jX_eY_2-Y_3}{(Y_2^2+Y_3^2)(1-X_e^2)} \\ 0 & \frac{Y_3+jX_eY_2}{(Y_2^2+Y_3^2)(1-X_e^2)} & \frac{-jX_eY_3+Y_2}{(Y_2^2+Y_3^2)(1-X_e^2)} \end{bmatrix}, \tag{4.205}$$

$$A^{-1}M_h^{-1} = \begin{bmatrix} \frac{1}{Y_1} & 0 & 0 \\ 0 & \frac{Y_2+jX_hY_3}{(Y_2^2+Y_3^2)(1-X_h^2)} & \frac{jX_hY_2-Y_3}{(Y_2^2+Y_3^2)(1-X_h^2)} \\ 0 & \frac{Y_3-jX_hY_2}{(Y_2^2+Y_3^2)(1-X_h^2)} & \frac{jX_hY_3+Y_2}{(Y_2^2+Y_3^2)(1-X_h^2)} \end{bmatrix}. \tag{4.206}$$

4.4.1 Voigt ($\vec{k} \perp \vec{B}_0$) and Faraday ($\vec{k} \parallel \vec{B}_0$) Geometries for $\vec{B}_0$ Perpendicular to the Interface

In the previous section, $\tilde{\epsilon}_p(\omega)$ was derived for arbitrary wave vectors (k_y and k_z). When the permittivity tensor elements are substituted back into the general transcendental dispersion equation, it will be easier for a root finding algorithm to find the roots if the equation has only a single variable. Although root finding can be done for multiple variables, to visualize what propagating wave numbers (k_y and k_z) satisfy the dispersion equation, most of the works [29, 49, 50] consider the case of a single wave number which is governed by the orientation of the DC magnetic field $\vec{B}_0$. To get more insight into what modes can exist given a surface wave type field distribution, a similar approach will be undertaken. With $\vec{B}_0 = 0$, it was found that arbitrarily any wave number can be neglected. But for $\vec{B}_0 \neq 0$, Voigt or Faraday geometries can be considered to aid the investigation.

4.4.2 Voigt Geometry ($\vec{k} \perp \vec{B}_0$) with $\vec{B}_0$ Perpendicular to the Interface

Having found the permittivity tensor $\tilde{\epsilon}_p(\omega)$ for $\vec{B}_0 = \hat{x}B_0$, the dispersion relation can be found by using the analysis done before for $\nu << \omega$, since the form of the tensor is the same as before [65]. The only difference is that the elements have some additional terms owing to the fact that the damping has also been included. The results have been discussed in the previous section when damping was neglected.

4.4.3   Faraday Geometry ($\vec{k} \parallel \vec{B}_0$) with $\vec{B}_0$ Perpendicular to the Interface

This geometry is of no importance in the context of analyzing surface waves, since there are no traveling waves in the x direction [49].

4.5 $\vec{B}_0$ Parallel to the Interface

When $\vec{B}_0$ is parallel to the interface, i.e., $\theta = 90°$, similar simplifications as mentioned above can be made [64, 66]. Going back to the equation

$$\vec{J} = -N_0 q [I + \frac{j\nu_{eh}}{\omega} M_e^{-1} + \frac{j\nu_{he}}{\omega} M_h^{-1}]^{-1} (-\frac{jq}{\omega m_e} M_e^{-1} - \frac{jq}{\omega m_h} M_h^{-1}) \cdot \vec{E}, \qquad (4.207)$$

the general matrix inverses M_e^{-1} and M_h^{-1} have been derived before as

$$M_e^{-1} = \frac{1}{\Delta_e} \begin{bmatrix} 1 - X_e^2 cos^2\theta & -X_e^2 cos\theta sin\theta & jX_e sin\theta \\ -X_e^2 cos\theta sin\theta & 1 - X_e^2 sin^2\theta & -jX_e cos\theta \\ -jX_e sin\theta & jX_e cos\theta & 1 \end{bmatrix}, \qquad (4.208)$$

$$M_h^{-1} = \frac{1}{\Delta_h} \begin{bmatrix} 1 - X_h^2 cos^2\theta & -X_h^2 cos\theta sin\theta & -jX_h sin\theta \\ -X_h^2 cos\theta sin\theta & 1 - X_h^2 sin^2\theta & jX_h cos\theta \\ jX_h sin\theta & -jX_h cos\theta & 1 \end{bmatrix}, \qquad (4.209)$$

where again $\Delta_e = 1 - X_e^2$ and $\Delta_h = 1 - X_h^2$ are the determinants of M_e and M_h, respectively. Applying $\theta = 90°$, we get

$$M_e^{-1} = \frac{1}{\Delta_e} \begin{bmatrix} 1 & 0 & jX_e \\ 0 & 1 - X_e^2 & 0 \\ -jX_e & 0 & 1 \end{bmatrix}, \qquad (4.210)$$

$$M_h^{-1} = \frac{1}{\Delta_h} \begin{bmatrix} 1 & 0 & -jX_h \\ 0 & 1 - X_h^2 & 0 \\ jX_h & 0 & 1 \end{bmatrix}. \qquad (4.211)$$

The inverse of $A = I + \frac{j\nu_{eh}}{\omega} M_e^{-1} + \frac{j\nu_{he}}{\omega} M_h^{-1}$ is to be found first. The notations Γ_e and Γ_h

are used for the ratios $\frac{\nu_{eh}}{\omega}$ and $\frac{\nu_{he}}{\omega}$, respectively, and $X_e = \frac{\Omega_{ce}}{\omega}, X_h = \frac{\Omega_{ch}}{\omega}$. Therefore, A is given by

$$
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{j\Gamma_e}{\Delta_e} \begin{bmatrix} 1 & 0 & jX_e \\ 0 & 1 - X_e^2 & 0 \\ -jX_e & 0 & 1 \end{bmatrix} + \frac{j\Gamma_h}{\Delta_h} \begin{bmatrix} 1 & 0 & -jX_h \\ 0 & 1 - X_h^2 & 0 \\ jX_h & 0 & 1 \end{bmatrix}. \quad (4.212)
$$

Doing the required operations on A, we get

$$
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{j\Gamma_e}{1-X_e^2} & 0 & \frac{-X_e\Gamma_e}{1-X_e^2} \\ 0 & j\Gamma_e & 0 \\ \frac{X_e\Gamma_e}{1-X_e^2} & 0 & \frac{j\Gamma_e}{1-X_e^2} \end{bmatrix} + \begin{bmatrix} \frac{j\Gamma_h}{1-X_h^2} & 0 & \frac{X_h\Gamma_h}{1-X_h^2} \\ 0 & j\Gamma_h & 0 \\ \frac{-X_h\Gamma_h}{1-X_h^2} & 0 & \frac{j\Gamma_h}{1-X_h^2} \end{bmatrix}, \quad (4.213)
$$

or

$$
A = \begin{bmatrix} 1 + \frac{j\Gamma_e}{1-X_e^2} + \frac{j\Gamma_h}{1-X_h^2} & 0 & \frac{X_h\Gamma_h}{1-X_h^2} - \frac{X_e\Gamma_e}{1-X_e^2} \\ 0 & j(\Gamma_e + \Gamma_h) & 0 \\ \frac{X_e\Gamma_e}{1-X_e^2} - \frac{X_h\Gamma_h}{1-X_h^2} & 0 & 1 + \frac{j\Gamma_e}{1-X_e^2} + \frac{j\Gamma_h}{1-X_h^2} \end{bmatrix}. \quad (4.214)
$$

As done before, let $Y_1 = j(\Gamma_e + \Gamma_h)$, $Y_2 = 1 + \frac{j\Gamma_e}{1-X_e^2} + \frac{j\Gamma_h}{1-X_h^2}$, and $Y_3 = \frac{X_e\Gamma_e}{1-X_e^2} - \frac{X_h\Gamma_h}{1-X_h^2}$, to be consistent with $\theta = 0°$ case. The required A matrix that we needed is therefore given by

$$
A = \begin{bmatrix} Y_2 & 0 & -Y_3 \\ 0 & Y_1 & 0 \\ Y_3 & 0 & Y_2 \end{bmatrix}. \quad (4.215)
$$

Further, the inverse of A is given by

$$
A^{-1} = \frac{1}{\Delta_A} \begin{bmatrix} Y_1Y_2 & 0 & Y_1Y_3 \\ 0 & Y_2^2 + Y_3^2 & 0 \\ -Y_1Y_3 & 0 & Y_1Y_2 \end{bmatrix}, \quad (4.216)
$$

and $\Delta_A = Y_1(Y_2^2 + Y_3^2)$ is the determinant of matrix A. Simplifying eq. (4.216) further, we get

$$A^{-1} = \begin{bmatrix} \frac{Y_2}{Y_2^2+Y_3^2} & 0 & \frac{Y_3}{Y_2^2+Y_3^2} \\ 0 & \frac{1}{Y_1} & 0 \\ \frac{-Y_3}{Y_2^2+Y_3^2} & 0 & \frac{Y_2}{Y_2^2+Y_3^2} \end{bmatrix}. \tag{4.217}$$

Thus, $\vec{J}$ and $\vec{\nabla} \times \vec{H}$ equations can be written as

$$\vec{J} = (\frac{jN_0q^2}{\omega m_e} A^{-1} M_e^{-1} + \frac{jN_0q^2}{\omega m_h} A^{-1} M_h^{-1}) \cdot \vec{E}, \tag{4.218}$$

$$\vec{\nabla} \times \vec{H} = -j\omega\epsilon_0(I - (\frac{N_0q^2}{\omega^2 m_e \epsilon_0} A^{-1} M_e^{-1} + \frac{N_0q^2}{\omega^2 m_h \epsilon_0} A^{-1} M_h^{-1})) \cdot \vec{E}, \tag{4.219}$$

$$\Rightarrow \vec{\nabla} \times \vec{H} = -j\omega\epsilon_0(I - (\frac{\omega_{pe}^2}{\omega^2} A^{-1} M_e^{-1} + \frac{\omega_{ph}^2}{\omega^2} A^{-1} M_h^{-1})) \cdot \vec{E}. \tag{4.220}$$

The required permittivity tensor $\tilde{\epsilon}_p$ can be written as

$$\tilde{\epsilon}_p(\omega) = I - \frac{\omega_{pe}^2}{\omega^2} A^{-1} M_e^{-1} - \frac{\omega_{ph}^2}{\omega^2} A^{-1} M_h^{-1}, \tag{4.221}$$

which is the same expression as before, but $A^{-1} M_e^{-1}$ and $A^{-1} M_h^{-1}$ have different values owing to a different orientation of $\vec{B}_0$. These are given by

$$A^{-1} M_e^{-1} = \begin{bmatrix} \frac{Y_2 - jX_eY_3}{(1-X_e^2)(Y_2^2+Y_3^2)} & 0 & \frac{Y_3 + jX_eY_2}{(1-X_e^2)(Y_2^2+Y_3^2)} \\ 0 & \frac{1}{Y_1} & 0 \\ \frac{-Y_3 - jX_eY_2}{(1-X_e^2)(Y_2^2+Y_3^2)} & 0 & \frac{Y_2 - jX_eY_3}{(1-X_e^2)(Y_2^2+Y_3^2)} \end{bmatrix}, \tag{4.222}$$

$$A^{-1} M_h^{-1} = \begin{bmatrix} \frac{Y_2 + jX_hY_3}{(1-X_h^2)(Y_2^2+Y_3^2)} & 0 & \frac{Y_3 - jX_hY_2}{(1-X_h^2)(Y_2^2+Y_3^2)} \\ 0 & \frac{1}{Y_1} & 0 \\ \frac{-Y_3 + jX_hY_2}{(1-X_h^2)(Y_2^2+Y_3^2)} & 0 & \frac{Y_2 + jX_hY_3}{(1-X_h^2)(Y_2^2+Y_3^2)} \end{bmatrix}. \tag{4.223}$$

It is evident that the form of the permittivity tensor is the same as for the case when $\nu << \omega$. So, the dispersion relations will be the same apart from the tensor elements, which will have some more information about the semiconductor substrate [29, 49, 50, 64–66].

4.5.1 Results for Voigt Geometry with B_0 Parallel to the Interface

As mentioned before, this case is a degenerate case since the wave equation inside the substrate reduces to one with two field quantities being independent of the other one. The same cases as studied for when $\vec{B}_0$ was perpendicular to the interface, are analyzed. Effects of damping and magnetic field strengths are discussed below.

Firstly, when damping is included, for any strength of the static magnetic field, the cyclotron resonances are damped out for a nonzero collision frequency. This is shown in fig. 4.7, where the arrows indicate Ω_{ce} and Ω_{ch} and the anti-resonance is more pronounced in this case compared to that of B_0 perpendicular to the interface. Secondly, there exists no cutoff in the lower frequency region for any value of static magnetic field as shown in fig. 4.8, which was the case with B_0 perpendicular to the interface [46]. Plots for different strength of magnetic field is also shown in fig. 4.9.

4.5.2 Results for Faraday Geometry with B_0 Parallel to the Interface

In this section, the plots of dispersion relation with different magnitudes of B_0 and effect of damping are discussed. A plot which compares the effect of damping is shown in fig. 4.10. It might seem that there are many surface wave poles for both damped and undamped dispersion relations, but when the figure is zoomed into, one finds that not only are the poles for the two cases shifted, there are only few frequencies at which the dispersion relation value goes to a sufficient minimum. This can be seen in fig. 4.11, shown by arrows. There are also issues with the resolution, since the function value does not go to a number which is low enough to be called a minimum. But when the same case is run at a higher resolution, a more accurate pole can be found.

When the strength of the static magnetic field is varied, one can see a sharp peak at Ω_{ch} for $B_0 = 5T$, which does not happen when $B_0 = .05T$, even if zoomed in pretty well, as shown in fig. 4.12. Also, the low frequency cutoff after which the surface poles start, is absent, as was the case with B_0 perpendicular to the interface [12]. The plots for dispersion curves with different strength of B_0 is shown in fig. 4.13, which were run for a high resolution and were plotted on a semilogy scale.
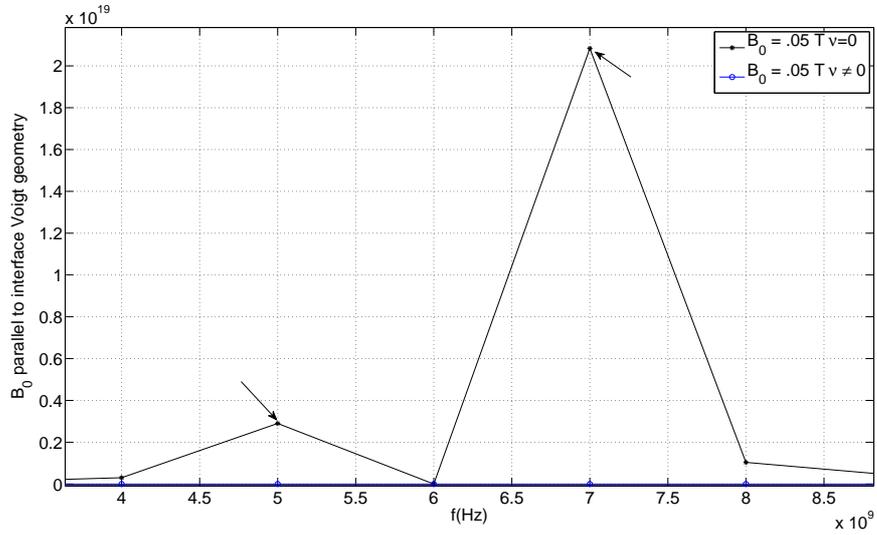
Fig. 4.7: Effect of damping on cyclotron resonances for $B_0 = .05T$, Voigt geometry.
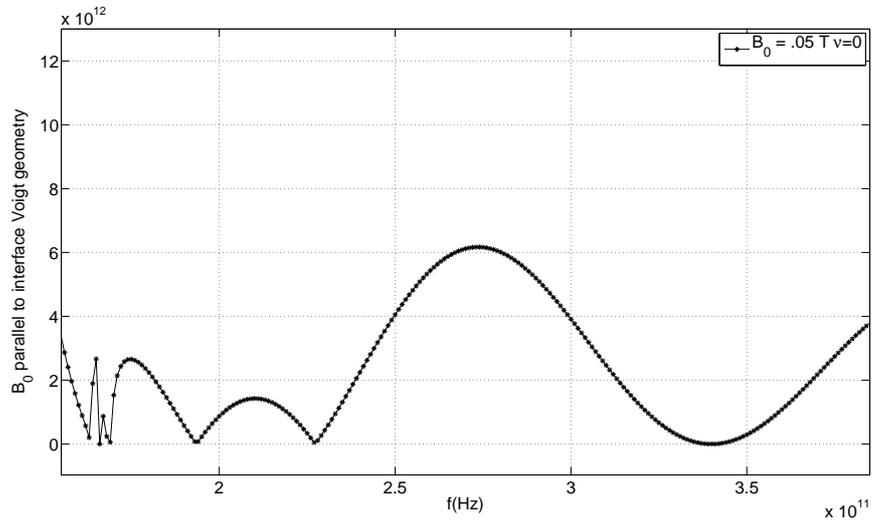
Fig. 4.8: Plot for $B_0 = .05T$ to show the absence of low frequency cutoff, Voigt geometry.
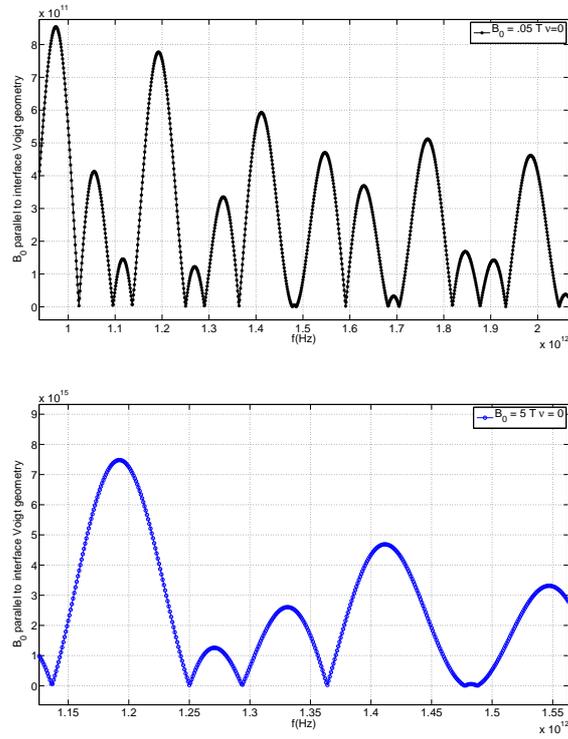
Fig. 4.9: Dispersion relation for different strengths of B_0, Voigt geometry.
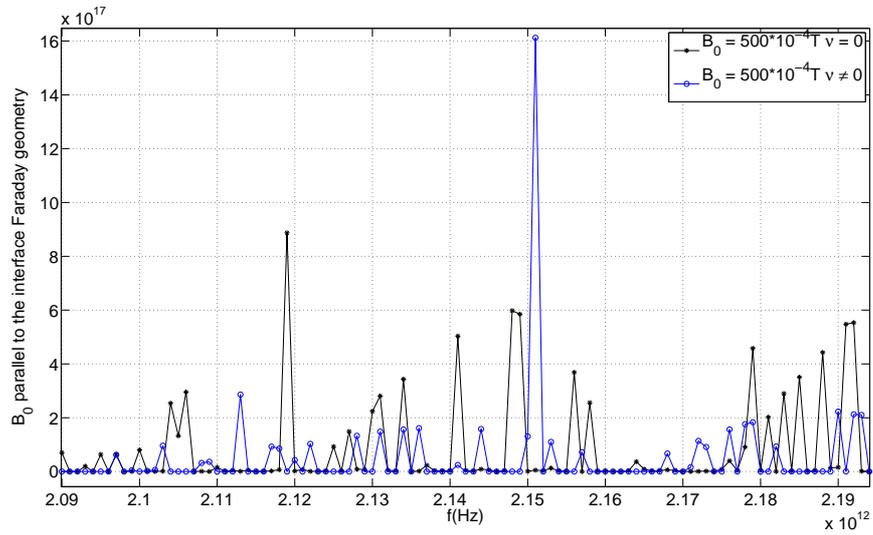
Fig. 4.10: Faraday geometry, effect of damping with $B_0 = .05T$.
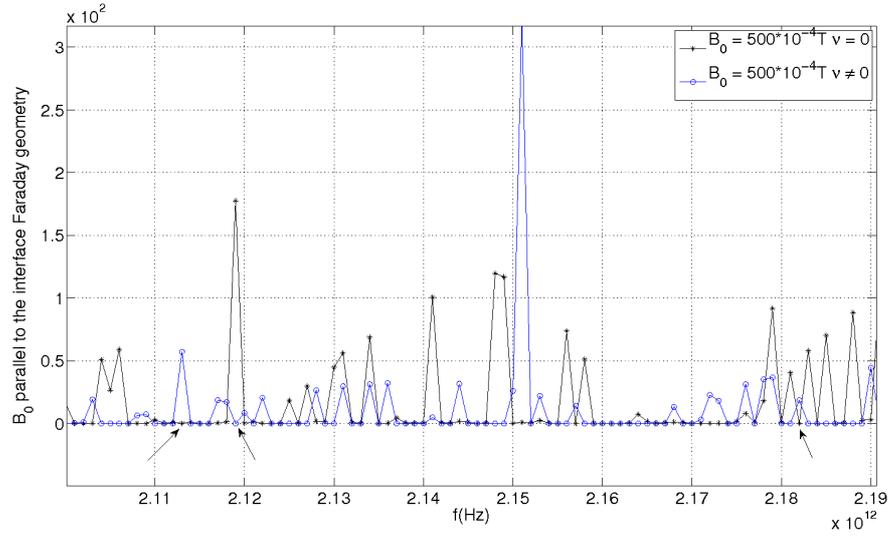
Fig. 4.11: Faraday geometry, effect of damping with $B_0 = .05T$ zoomed in.
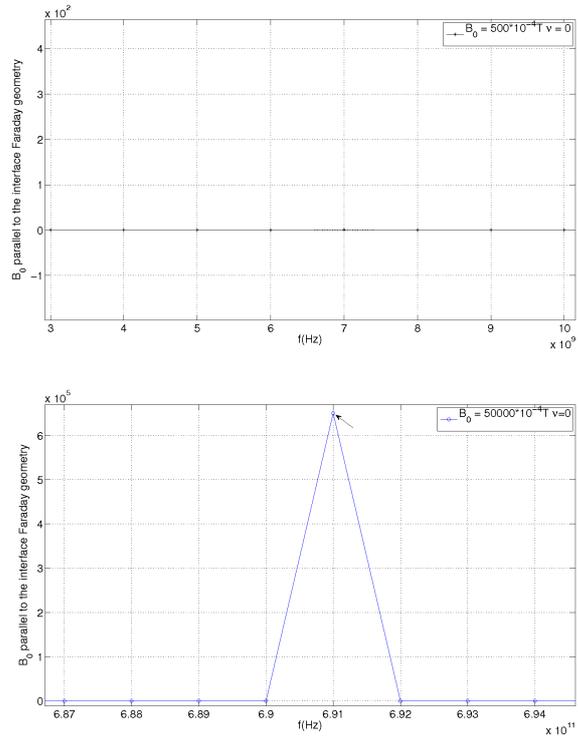
Fig. 4.12: Cyclotron resonance Ω_{ch} for $B_0 = 5T$.
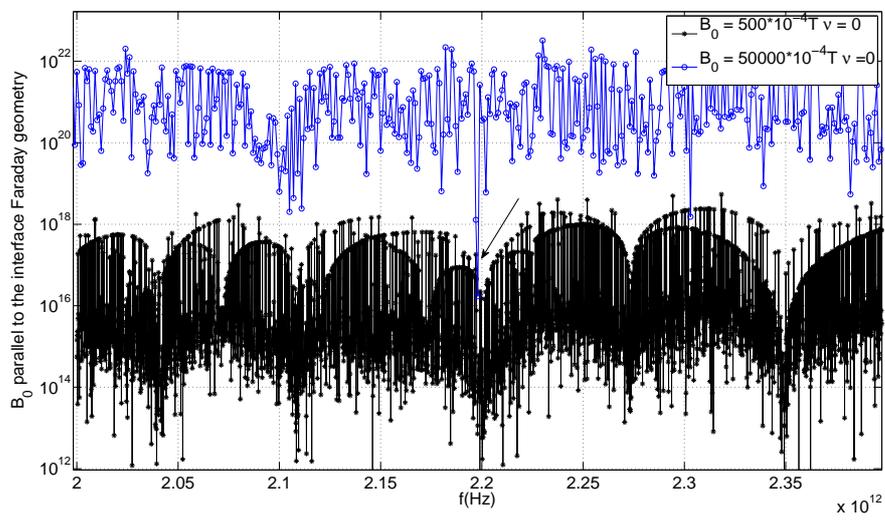
Fig. 4.13: Plots for $B_0 = .05T$ and $B_0 = 5T$ zoomed in.

Chapter 5

Spatial Dispersion in Semiconductor Medium

Spatial dispersion in a semiconductor can occur if there are density perturbations or background flow, i.e., $\vec{U}_0 \neq 0$ (implying a DC current density [32, 63]) in the medium. Background flow is generally important when dealing with space plasmas [63] and for semiconductor devices, such flows can generally be neglected. During this chapter, therefore, changes in density that cause the permittivity tensor $\tilde{\epsilon}_p$ to be a function of spatial coordinates along with ω, will be investigated. $\tilde{\epsilon}_p$, being a function of spatial frequency, will be a nonlocal function of space coordinates. Mathematically, spatial dispersion is accounted for by including the $\vec{\nabla} p$ term in the fluid momentum equation [64, 65] of electrons and holes. As mentioned in the first two chapters, this would require us to consider the fluid continuity equation as well as a closure equation [66], so that both the fluid and the Maxwell's curl equations are sufficient to describe the medium [38]. Whenever a plasma like medium such as a semiconductor is considered and a closure is needed on the system, a model for the medium has to be assumed. Therefore, either a degenerate gas model or a thermal model can be used for the closure equation. For a semiconductor, the model mostly used is the degenerate gas model [32].

Before going further into the details of the derivation, an analogy between temporal and spatial dispersion is needed that would be useful in understanding the dynamics of the medium. This has already been discussed in the introductory chapter. While deriving the tensor elements, first the $\vec{B}_0 = 0$ case would be considered and then a thorough investigation will be done when the steady magnetic field is turned on. Also, during the derivation, many important concepts will be reiterated so that spatial dispersion is clearly understood.

5.1 $\tilde{\epsilon}_p$ Derivation for $\vec{B}_0 = 0$

For this case, momentum and continuity equations will be needed for both the species, viz. holes and electrons. Since in this case the $\vec{\nabla}p$ term will be included, the derivation will start with the most general form of the two equations after which perturbation theory will be applied to linearize the equations. This will finally give the required tensor. The general momentum and continuity equation for a species s is given by [66]

$$\rho_s(\frac{\partial}{\partial t} + \vec{U}_s \cdot \vec{\nabla})\vec{U}_s = Q_s(\vec{E} + \vec{U}_s \times \vec{B}) - \vec{\nabla}P_s + \rho_s \sum_r \nu_{sr}(\vec{U}_r - \vec{U}_s), \tag{5.1}$$

$$\frac{\partial N_s}{\partial t} + \vec{\nabla} \cdot (N_s\vec{U}_s) = P - L, \tag{5.2}$$

$$\vec{J} = \sum_s N_s q_s \vec{U}_s, \tag{5.3}$$

and the pressure closure equation, for a degenerate gas model by

$$P_s = \frac{m_s v_{fs}^2 N_s}{3}, \tag{5.4}$$

$\vec{J}$ being the current density due to different species traveling at different perturbed velocities and v_{fs} is the Fermi velocity of species s. P and L are production and loss mechanisms and will be assumed to be zero, since they have negligible part to play in the behavior of surface wave modes [60]. Also,

$$\rho_s = m_s N_s, \tag{5.5}$$

$$Q_s = q_s N_s, \tag{5.6}$$

where m_s is the effective mass of either electron or hole and q_s is the charge on the species [66]. The Maxwell's curl equations should also be mentioned to complete the set of equations needed to solve the problem. They are

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{5.7}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}. \tag{5.8}$$

The equations are then linearized through the perturbation technique [2, 66], in which every field quantity as well as number density, flow velocity, etc., are split into large DC unperturbed values and small AC perturbations, that depend on time, i.e.,

$$Z = Z_0 + ze^{-j\omega t}, \tag{5.9}$$

where the lower case alphabets will denote the perturbed quantities (except for electric and magnetic fields, where for electric field and current density only $\vec{E}$ and $\vec{J}$ would be used denoting the perturbed quantity and for magnetic field, $\vec{B}_0$ and $\vec{B}$ will be used for unperturbed and perturbed quantities, respectively). The equations reduce to the following linear form, under zeroth order (and assuming $\vec{U}_{0s}$ and $\vec{E}_0$ to be zero, i.e., no bulk flows and no background electric field [65]),

$$\frac{\partial N_0}{\partial t} = 0, \tag{5.10}$$

$$\Rightarrow N_0 = constant, \tag{5.11}$$

$$0 = 0, \tag{5.12}$$

according to our assumption mentioned above.

The first order perturbations, through which the substrate permittivity will be derived, for a given species s is given as

$$\frac{\partial(n_s)}{\partial t} + \vec{\nabla} \cdot (N_{0s} + n_s)(\vec{U}_0 + \vec{u}_s) = R_s, \tag{5.13}$$

$$m_s(N_0 + n_s)(\frac{\partial}{\partial t} + \vec{u}_s \cdot \vec{\nabla})\vec{u}_s = q(N_0 + n_s)(\vec{E} + \vec{u}_s \times (\vec{B}_0 + \vec{B})) - \vec{\nabla}p$$
$$+ m(N_0 + n_s)\sum_j \nu_{ij}(\vec{u}_j - \vec{u}_i), \tag{5.14}$$

$$\Rightarrow -j\omega m_s N_0 \vec{u}_s = qN_0\vec{E} - \vec{\nabla}p$$
$$+ m(N_0 + n_s)\sum_j \nu_{ij}(\vec{u}_j - \vec{u}_i), \tag{5.15}$$

which simplify into

$$-j\omega n_s + \vec{\nabla} \cdot (N_0 \vec{u}_s) = R_s, \tag{5.16}$$

$$-j\omega \vec{u}_s = \frac{q}{m_s}\vec{E} - \frac{1}{m_s N_0}\vec{\nabla}p + mN_0 \sum_j \nu_{ij}(\vec{u}_j - \vec{u}_i), \tag{5.17}$$

$$\nabla \times \vec{E} = j\omega \vec{B}, \tag{5.18}$$

$$\nabla \times \vec{H} = \vec{J} - j\omega \epsilon_0 \vec{E}, \tag{5.19}$$

$$p_s = \frac{m_s v_{fs}^2 n_s}{3}. \tag{5.20}$$

Note that the constitutive relation, viz. $\vec{B} = \mu_0 \vec{H}$ [8, 10] has been used and it has been assumed that the material is nonpermeable. When a relation between the perturbed flow velocity and electric field is derived, using the relation $\vec{J} = Nq \sum \vec{U}$, we can apply perturbation theory and get the required relation between $\vec{J}$ and $\vec{E}$, noting that the background velocity $\vec{U}_0$ is zero. Therefore,

$$\vec{J} = N_0 q(\vec{u}_h - \vec{u}_e). \tag{5.21}$$

Equation (5.21) is valid for the case when the semiconductor is quasi-neutral (i.e., the background density for holes and electrons is equal). $\vec{u}_h$ and $\vec{u}_e$ denote the perturbed hole and electron flow velocities. The negative sign in front of the electron velocity is because of the negative charge on an electron. For the two species, i.e., holes and electrons, we get

$$\vec{u}_h = \frac{jq}{m_h\omega}\vec{E} - \frac{j}{\omega m_h N_0}\vec{\nabla}p_h + \frac{j\nu_{he}}{\omega}(\vec{u}_e - \vec{u}_h), \tag{5.22}$$

$$\vec{u}_e = -\frac{jq}{m_e\omega}\vec{E} - \frac{j}{\omega m_e N_0}\vec{\nabla}p_e + \frac{j\nu_{eh}}{\omega}(\vec{u}_h - \vec{u}_e), \tag{5.23}$$

where p_e and p_h are the electron and hole pressure terms. As such, there is no direct relation between pressure gradient $\vec{\nabla}p_s$ and fluid velocity $\vec{u}_s$. Some sort of equation generally called a closure equation is needed which relates pressure to perturbed density. Before writing a closure equation, we need to assume what type of gas/fluid a semiconductor is. Either it can be assumed a classical gas in which case the universal gas law would be the closure equation and the we would define thermal velocity or if it is assumed a degenerate gas, Fermi

statistics would give us the required relation and Fermi velocity would be defined [32, 66]. In most of the works, it is assumed that a semiconductor is a degenerate gas for which the following equations hold. For such a gas, the closure equation is given by $p_s = \frac{m_s v_{fs}^2 n_s}{3}$, where v_{fs} is the Fermi velocity given by $v_{fs} = \frac{h}{2\pi m_s}(3\pi^2 N_0)^{\frac{1}{3}}$, where h is the Planck's constant [32].

Our aim is to get an equation that relates $\vec{u}_h - \vec{u}_e$ and $\vec{E}$. The hole and electron continuity equations, i.e.,

$$-j\omega\vec{u}_h + N_0\vec{\nabla} \cdot u_h = R_h, \tag{5.24}$$

$$-j\omega\vec{u}_e + N_0\vec{\nabla} \cdot u_e = R_e, \tag{5.25}$$

must be employed. R_s is the net recombination rate for holes or electrons and will be assumed zero in the final steps. This rate is related to the perturbed density n_s through the corresponding lifetimes t_s, from Shockley-Read-Hall statistics and recombination theory [57]. In other words,

$$R_e = \frac{n_e}{t_e}, \tag{5.26}$$

$$R_h = \frac{n_h}{t_h}. \tag{5.27}$$

Before starting with the derivation of $\tilde{\epsilon}_p$, it should be understood that the assumptions made on the wave vectors with regards to the pressure term are separate from that of the wave equation. In other words, the modelling used for the derivation has nothing to do with the final wave equation. This is a valid statement, since the phase velocity of the electromagnetic waves is very much greater than the Fermi velocity. In other words, the electromagnetic waves travel at a phase velocity which is close to the speed of light and the waves due to density perturbations travel at a much lower velocity [64, 65]. Hence, the wave vector assumptions made during the derivation of $\tilde{\epsilon}_p$ has no bearing on the way the wave equations are written.

Starting with the momentum equation, we get

$$\vec{u}_h = \frac{jq}{m_h\omega}\vec{E} - \frac{jm_h v_{fh}^2}{3\omega m_h N_0}\vec{\nabla}n_h + \frac{j\nu_{he}}{\omega}(\vec{u}_e - \vec{u}_h), \tag{5.28}$$

$$\vec{u}_e = -\frac{jq}{m_e\omega}\vec{E} - \frac{jm_e v_{fe}^2}{3\omega m_e N_0}\vec{\nabla}n_e + \frac{j\nu_{eh}}{\omega}(\vec{u}_h - \vec{u}_e), \tag{5.29}$$

or

$$\vec{u}_h = \frac{jq}{m_h\omega}\vec{E} - \frac{jv_{fh}^2}{3\omega N_0}\vec{\nabla}n_h + \frac{j\nu_{he}}{\omega}(\vec{u}_e - \vec{u}_h), \tag{5.30}$$

$$\vec{u}_e = -\frac{jq}{m_e\omega}\vec{E} - \frac{jv_{fe}^2}{3\omega N_0}\vec{\nabla}n_e + \frac{j\nu_{eh}}{\omega}(\vec{u}_h - \vec{u}_e), \tag{5.31}$$

and from eqs. (5.24) and (5.25), we can substitute for $\vec{\nabla}n_s$ in terms of the fluid velocity $\vec{u}_s$ as

$$-j\omega n_h + \vec{\nabla}\cdot(N_0\vec{u}_h) = \frac{n_h}{t_h}, \tag{5.32}$$

$$-j\omega n_e + \vec{\nabla}\cdot(N_0\vec{u}_e) = \frac{n_e}{t_e}, \tag{5.33}$$

or simply

$$n_h = \frac{N_0}{j\omega + \frac{1}{t_h}}\vec{\nabla}\cdot\vec{u}_h, \tag{5.34}$$

$$n_e = \frac{N_0}{j\omega + \frac{1}{t_e}}\vec{\nabla}\cdot\vec{u}_e. \tag{5.35}$$

Equations (5.28) and (5.29) can then be rewritten as

$$\vec{u}_h = \frac{jq}{m_h\omega}\vec{E} - \frac{jv_{fh}^2}{3\omega(j\omega + \frac{1}{t_h})}\vec{\nabla}(\vec{\nabla}\cdot\vec{u}_h) + \frac{j\nu_{he}}{\omega}(\vec{u}_e - \vec{u}_h), \tag{5.36}$$

$$\vec{u}_e = -\frac{jq}{m_e\omega}\vec{E} - \frac{jv_{fe}^2}{3\omega(j\omega + \frac{1}{t_e})}\vec{\nabla}(\vec{\nabla}\cdot\vec{u}_e) + \frac{j\nu_{eh}}{\omega}(\vec{u}_h - \vec{u}_e). \tag{5.37}$$

We need to put some constraint on the $\vec{\nabla}(\vec{\nabla}\cdot\vec{u}_h)$ term so that $\tilde{\epsilon}_p$ is a manageable function of wave number. Firstly, we assume that the perturbed density only varies along

the direction perpendicular to the interface, i.e., $\frac{\partial}{\partial y} = 0$ and $\frac{\partial}{\partial z} = 0$ [47]. This makes the $\vec{\nabla}$ operator in spatial frequency domain as $\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} = \hat{x}jk_x$. Taking these assumptions into account, the hole and electron momentum equations will become algebraic in nature that are easy to handle as far as finding a relation between $\vec{u}_h - \vec{u}_e$ and $\vec{E}$ is concerned. These can be written as

$$\vec{u}_e = -\frac{jq}{m_e\omega}\vec{E} - \frac{j\beta_e^2}{\omega(j\omega + \frac{1}{t_e})}(\hat{x}jk_x)(jk_x u_{ex}) + \frac{j\nu_{eh}}{\omega}(\vec{u}_h - \vec{u}_e), \tag{5.38}$$

$$\vec{u}_h = \frac{jq}{m_h\omega}\vec{E} - \frac{j\beta_h^2}{\omega(j\omega + \frac{1}{t_h})}(\hat{x}jk_x)(jk_x u_{hx}) + \frac{j\nu_{he}}{\omega}(\vec{u}_e - \vec{u}_h), \tag{5.39}$$

or

$$\vec{u}_e = -\frac{jq}{m_e\omega}\vec{E} + \frac{j\beta_e^2}{\omega(j\omega + \frac{1}{t_e})}(\hat{x}k_x^2 u_{ex}) + \frac{j\nu_{eh}}{\omega}(\vec{u}_h - \vec{u}_e), \tag{5.40}$$

$$\vec{u}_h = \frac{jq}{m_h\omega}\vec{E} + \frac{j\beta_h^2}{\omega(j\omega + \frac{1}{t_h})}(\hat{x}k_x^2 u_{hx}) + \frac{j\nu_{he}}{\omega}(\vec{u}_e - \vec{u}_h), \tag{5.41}$$

where $\beta_s^2 = \frac{v_{fe}^2}{3}$. Even though $\vec{B}_0 = 0$, spatial dispersion ($\vec{\nabla}p_s$) can cause anisotropy in the ϵ_p tensor, which was absent previously when $\vec{\nabla}p_s$ was neglected. Writing these equations in component form, we get

$$\hat{x} \Rightarrow u_{ex} = -\frac{jq}{m_e\omega}E_x + \frac{j\beta_e^2}{\omega(j\omega + \frac{1}{t_e})}(k_x^2 u_{ex}) + \frac{j\nu_{eh}}{\omega}(u_{hx} - u_{ex}), \tag{5.42}$$

$$\hat{y} \Rightarrow u_{ey} = -\frac{jq}{m_e\omega}E_y + \frac{j\nu_{eh}}{\omega}(u_{hy} - u_{ey}), \tag{5.43}$$

$$\hat{z} \Rightarrow u_{ez} = -\frac{jq}{m_e\omega}E_z + \frac{j\nu_{eh}}{\omega}(u_{hz} - u_{ez}), \tag{5.44}$$

or in matrix form as

$$\begin{bmatrix} 1 - \frac{j\beta_e^2 k_x^2}{\omega(j\omega + \frac{1}{t_e})} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{ex} \\ u_{ey} \\ u_{ez} \end{bmatrix} = -\frac{jq}{m_e\omega}\vec{E} + \frac{j\nu_{eh}}{\omega}(\vec{u}_h - \vec{u}_e). \tag{5.45}$$

Similarly for holes, through the same procedure, we get

$$
\begin{bmatrix} 1 - \frac{j\beta_h^2 k_x^2}{\omega(j\omega + \frac{1}{t_h})} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{hx} \\ u_{hy} \\ u_{hz} \end{bmatrix} = \frac{jq}{m_h\omega}\vec{E} + \frac{j\nu_{he}}{\omega}(\vec{u}_e - \vec{u}_h). \tag{5.46}
$$

In compact form, eqs. (5.45) and (5.46) can be written as

$$
M_e \cdot \vec{u}_e = -\frac{jq}{m_e\omega}\vec{E} + \frac{j\nu_{eh}}{\omega}(\vec{u}_h - \vec{u}_e), \tag{5.47}
$$

$$
M_h \cdot \vec{u}_h = \frac{jq}{m_h\omega}\vec{E} + \frac{j\nu_{he}}{\omega}(\vec{u}_e - \vec{u}_h), \tag{5.48}
$$

where

$$
M_e = \begin{bmatrix} 1 - \frac{j\beta_e^2 k_x^2}{\omega(j\omega + \frac{1}{t_e})} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5.49}
$$

and

$$
M_h = \begin{bmatrix} 1 - \frac{j\beta_h^2 k_x^2}{\omega(j\omega + \frac{1}{t_h})} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.50}
$$

Finally, multiplying the eqs. (5.47) and (5.48) with the respective inverses of M_e and M_h, we can readily subtract

$$
\vec{u}_e = -\frac{jq}{m_e\omega}M_e^{-1} \cdot \vec{E} + \frac{j\nu_{eh}}{\omega}M_e^{-1} \cdot (\vec{u}_h - \vec{u}_e), \tag{5.51}
$$

$$
\vec{u}_h = \frac{jq}{m_h\omega}M_h^{-1} \cdot \vec{E} + \frac{j\nu_{he}}{\omega}M_h^{-1} \cdot (\vec{u}_e - \vec{u}_h), \tag{5.52}
$$

to get the required relation between $\vec{u}_h - \vec{u}_e$ and $\vec{E}$ as

$$
[I + \frac{j\nu_{eh}}{\omega}M_e^{-1} + \frac{j\nu_{he}}{\omega}M_h^{-1}] \cdot (\vec{u}_h - \vec{u}_e) = (\frac{jq}{m_e\omega}M_e^{-1} + \frac{jq}{m_h\omega}M_h^{-1}) \cdot \vec{E}, \tag{5.53}
$$

or

$$\vec{u}_h - \vec{u}_e = [I + \frac{j\nu_{eh}}{\omega}M_e^{-1} + \frac{j\nu_{he}}{\omega}M_h^{-1}]^{-1}(\frac{jq}{m_e\omega}M_e^{-1} + \frac{jq}{m_h\omega}M_h^{-1}) \cdot \vec{E}. \qquad (5.54)$$

Substituting eq. (5.54) back into the current density equation, we get

$$\vec{J} = N_0 q[I + \frac{j\nu_{eh}}{\omega}M_e^{-1} + \frac{j\nu_{he}}{\omega}M_h^{-1}]^{-1}(\frac{jq}{m_e\omega}M_e^{-1} + \frac{jq}{m_h\omega}M_h^{-1}) \cdot \vec{E}, \qquad (5.55)$$

where such a relation can be written only when one is in both spatial and temporal frequency domain. The Maxwell's $\vec{H}$ curl equation can then be made homogeneous by using eq. (5.55) and we get a tensor for ϵ_p as

$$\vec{\nabla} \times \vec{H} = N_0 q[I + \frac{j\nu_{eh}}{\omega}M_e^{-1} + \frac{j\nu_{he}}{\omega}M_h^{-1}]^{-1}(\frac{jq}{m_e\omega}M_e^{-1} + \frac{jq}{m_h\omega}M_h^{-1}) \cdot \vec{E}$$
$$-j\omega\epsilon_0\vec{E}, \qquad (5.56)$$
$$\Rightarrow \vec{\nabla} \times \vec{H} = -j\omega\epsilon_0(I - [I + \frac{j\nu_{eh}}{\omega}M_e^{-1} + \frac{j\nu_{he}}{\omega}M_h^{-1}]^{-1}(\frac{\omega_{pe}^2}{\omega^2}M_e^{-1}$$
$$+\frac{\omega_{ph}^2}{\omega^2}M_h^{-1})) \cdot \vec{E}, \qquad (5.57)$$

where ω_{pe} and ω_{ph} are the respective plasma frequencies for electrons and holes [64–66]. The permittivity tensor can then be written as

$$\tilde{\epsilon}_p = I - [I + \frac{j\nu_{eh}}{\omega}M_e^{-1} + \frac{j\nu_{he}}{\omega}M_h^{-1}]^{-1}(\frac{\omega_{pe}^2}{\omega^2}M_e^{-1} + \frac{\omega_{ph}^2}{\omega^2}M_h^{-1}). \qquad (5.58)$$

Although, the steady magnetic field is absent, the semiconductor permittivity is still a tensor. In other words, density perturbations can cause the semiconductor medium to behave anisotropically. To get a solution for the dispersion relation given that ϵ_p has the above form can still have issues since both k_y and k_z are nonzero. In order that a root-finding algorithm be applied to such an equation, either of the wave numbers have to be assumed zero. Also throughout the chapter, it will be assumed that the recombination time, i.e., $t_s = \infty$ [50]. This makes the matrices a lot simpler.

Derivation of $\tilde{\epsilon}_p$ requires M_e^{-1}, M_h^{-1}, $A = I + \frac{j\nu_{eh}}{\omega}M_e^{-1} + \frac{j\nu_{he}}{\omega}M_h^{-1}$, and A^{-1}. Since M_e and M_h are given by

$$M_e = \begin{bmatrix} 1 - \frac{\beta_e^2 k_x^2}{\omega^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5.59}$$

$$M_h = \begin{bmatrix} 1 - \frac{\beta_h^2 k_x^2}{\omega^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.60}$$

Then, M_s^{-1} is

$$M_s^{-1} = \frac{1}{\Delta_s} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 - \frac{\beta_s^2}{\omega^2}k_x^2 & 0 \\ 0 & 0 & 1 - \frac{\beta_s^2}{\omega^2}k_x^2 \end{bmatrix}, \tag{5.61}$$

$$\Rightarrow M_s^{-1} = \begin{bmatrix} \frac{1}{1 - \frac{\beta_s^2}{\omega^2}k_x^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5.62}$$

since $\Delta_s = 1 - \frac{\beta_s^2}{\omega^2}k_x^2$.

Next, $A = I + \frac{j\nu_{eh}}{\omega}M_e^{-1} + \frac{j\nu_{he}}{\omega}M_h^{-1} = I + j\Gamma_e M_e^{-1} + j\Gamma_h M_h^{-1}$ can be calculated as being equal to

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{j\Gamma_e}{1 - \frac{\beta_e^2}{\omega^2}k_x^2} & 0 & 0 \\ 0 & j\Gamma_e & 0 \\ 0 & 0 & j\Gamma_e \end{bmatrix} + \begin{bmatrix} \frac{j\Gamma_h}{1 - \frac{\beta_h^2}{\omega^2}k_x^2} & 0 & 0 \\ 0 & j\Gamma_h & 0 \\ 0 & 0 & j\Gamma_h \end{bmatrix}. \tag{5.63}$$

Before adding the matrices, one needs to get a permittivity model similar to that of Agarwal et al. [47], so that the finite spatial Fourier transform in the x direction can be manipulated without doing the integration itself. To do that, we need to unify in some sense, the hole

and electron Fermi velocities, i.e., $v_{fs} = \frac{h}{2\pi m_s}3\pi^2 N_0$ [32]. Since these velocities depend on the effective masses of holes and electrons, which in turn are $m_e = 2.3665 \times 10^{-31}$ Kgs and $m_h = 1.5764 \times 10^{-31}$ Kgs [60], i.e., a difference of two units approximately on a scale of 10^{-31}, they can be assumed equal. This implies that $\beta_e \simeq \beta_h \simeq \beta_T$, which is called the thermal β of the semiconductor medium [32]. Note that the x dependence in medium 2, i.e., $e^{-\beta x}$, is completely different from this thermal β and should not be confused with it.

Using the above simplification, the matrix A can then be written as

$$A = \begin{bmatrix} 1 + \frac{j(\Gamma_e + \Gamma_h)}{1 - \frac{\beta_T^2}{\omega^2}k_x^2} & 0 & 0 \\ 0 & 1 + j(\Gamma_e + \Gamma_h) & 0 \\ 0 & 0 & 1 + j(\Gamma_e + \Gamma_h) \end{bmatrix}. \tag{5.64}$$

Note that the assumption of the effective masses being comparable has only been considered for the Fermi velocities and the plasma frequencies of holes and electrons are still unequal. We now need to invert this matrix and multiply the result with M_e^{-1} and M_h^{-1}. A^{-1} is given as

$$A^{-1} = \begin{bmatrix} \frac{1}{1 + \frac{j(\Gamma_e + \Gamma_h)}{1 - \frac{\beta_T^2}{\omega^2}k_x^2}} & 0 & 0 \\ 0 & 1 + j(\Gamma_e + \Gamma_h) & 0 \\ 0 & 0 & 1 + j(\Gamma_e + \Gamma_h) \end{bmatrix}, \tag{5.65}$$

which in turn gives

$$\frac{\omega_{pe}^2}{\omega^2}A^{-1}M_e^{-1} = \begin{bmatrix} \frac{\frac{\omega_{pe}^2}{\omega^2}}{1 + j(\Gamma_e + \Gamma_h) - \frac{\beta_T^2}{\omega^2}k_x^2} & 0 & 0 \\ 0 & \frac{\omega_{pe}^2}{\omega^2}(1 + j(\Gamma_e + \Gamma_h)) & 0 \\ 0 & 0 & \frac{\omega_{pe}^2}{\omega^2}(1 + j(\Gamma_e + \Gamma_h)) \end{bmatrix}. \tag{5.66}$$

Similarily for holes, we get

$$\frac{\omega_{ph}^2}{\omega^2} A^{-1} M_h^{-1} = \begin{bmatrix} \dfrac{\frac{\omega_{ph}^2}{\omega^2}}{1+j(\Gamma_e+\Gamma_h)-\frac{\beta_T^2}{\omega^2}k_x^2} & 0 & 0 \\ 0 & \frac{\omega_{ph}^2}{\omega^2}(1+j(\Gamma_e+\Gamma_h)) & 0 \\ 0 & 0 & \frac{\omega_{ph}^2}{\omega^2}(1+j(\Gamma_e+\Gamma_h)) \end{bmatrix}. \quad (5.67)$$

Thus,

$$\tilde{\epsilon}_p(\vec{k};\omega) = I - \frac{\omega_{pe}^2}{\omega^2} A^{-1} M_e^{-1} - \frac{\omega_{ph}^2}{\omega^2} A^{-1} M_h^{-1} \quad (5.68)$$

can be reduced using the above matrices as

$$\tilde{\epsilon}_p(\vec{k},\omega) = \begin{bmatrix} 1 - \dfrac{\frac{\omega_{pe}^2+\omega_{ph}^2}{\omega^2}}{1+j(\Gamma_e+\Gamma_h)-\frac{\beta_T^2}{\omega^2}k_x^2} & 0 & 0 \\ 0 & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{yy} \end{bmatrix}, \quad (5.69)$$

where the first element of the matrix (ϵ_{xx}) has both the local and nonlocal behavior of the material, the rest elements being independent of the wave vector. Also,

$$\epsilon_{yy} = 1 - \frac{\omega_{pe}^2 + \omega_{ph}^2}{\omega^2}(1 + j(\Gamma_e + \Gamma_h)). \quad (5.70)$$

Therefore, the wave equation will be in integro-differential form only for the x component [31, 32, 47]. The rest of the components have the same permittivity as derived for the isotropic case with $\vec{\nabla}p_s = 0$. $\tilde{\epsilon}_p$, although a diagonal matrix is not isotropic, but is uniaxially anisotropic [53], due to spatial dispersion. For a permittivity matrix (eq. (5.69)) to be analytically feasible, one needs to inverse Fourier transform ϵ_{xx}, so that a Green's function form in space coordinates can be derived and appropriate manipulations be done to render

a differential equation in x. ϵ_{xx} is given as

$$\epsilon_{xx}(k_x;\omega) = 1 - \frac{\frac{\omega_{pe}^2+\omega_{ph}^2}{\omega^2}}{1 + j(\Gamma_e + \Gamma_h) - \frac{\beta_T^2}{\omega^2}k_x^2}, \tag{5.71}$$

or after simplifying, as

$$\epsilon_{xx}(k_x;\omega) = 1 + \frac{\frac{\omega_{pe}^2+\omega_{ph}^2}{\beta_T^2}}{k_x^2 - \frac{\omega^2}{\beta_T^2}(1 + j(\Gamma_e + \Gamma_h))}, \tag{5.72}$$

$$\Rightarrow \epsilon_{xx}(k_x;\omega) = 1 + \frac{\chi}{k_x^2 - \gamma^2}, \tag{5.73}$$

where $\chi = \frac{\omega_{pe}^2+\omega_{ph}^2}{\beta_T^2}$ and $\gamma^2 = \frac{\omega^2}{\beta_T^2}(1 + j(\Gamma_e + \Gamma_h))$. Taking the inverse spatial Fourier transform, we get

$$\hat{\epsilon}_{xx}(x;\omega) = \delta(x) + \frac{\chi}{2\pi}\frac{e^{j\gamma|x|}}{|x|}. \tag{5.74}$$

It was previously stated that the material is translationally invariant in directions parallel to the interface. This assumption is taken further and through the dielectric approximation (surface corrections to the dielectric tensor are negligible) [32], which says that $\epsilon_{xx}(x, x';\omega)$ can be written as $\epsilon_{xx}(|x - x'|;\omega)$. This assumption enables us to write ϵ_{xx} as

$$\hat{\epsilon}_{xx}(x, x';\omega) = \delta(x - x') + \frac{\chi}{2\pi}G_\gamma(|x - x'|), \tag{5.75}$$

where $G_\gamma(|x-x'|) = \frac{e^{j\gamma|x-x'|}}{|x-x'|}$ is the kernel, well-known as the free space Green's function [26]. Then, the permittivity tensor can be written as

$$\tilde{\epsilon}_p(|x - x';\omega|) = \begin{bmatrix} \delta(x - x') + \frac{\chi}{2\pi}G_\gamma(|x - x'|) & 0 & 0 \\ 0 & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{yy} \end{bmatrix}, \tag{5.76}$$

where

$$\epsilon_{yy} = (1 - \frac{\omega_{pe}^2 + \omega_{ph}^2}{\omega^2}(1 + j(\Gamma_e + \Gamma_h)))\delta(x - x'). \tag{5.77}$$

Also, ϵ_{yy} is a delta function of x coordinate, since the inverse Fourier transform of a constant in spatial frequency is the constant multiplied by delta function in spatial coordinates. These elements of are still local functions of spatial coordinates and so the y and z components of the wave equation will be of the same form as the isotropic case analyzed in the third chapter.

Writing the vector wave equation, we have

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E}(\vec{x}; \omega) = \omega^2 \mu_0 \epsilon_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dy' dz' \int_0^d dx' \tilde{\epsilon}_p(|x - x'|; \omega) \cdot \vec{E}(\vec{x}; \omega). \qquad (5.78)$$

To reduce the above differential equation to an algebraic equation, let the spatial Fourier transform be first defined for the infinite coordinates, in the form of a two-dimensional transform as [47]

$$\vec{E}(\vec{r}; \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{E}^*(x; \omega) e^{j(k_y y + k_z z)} dk_y dk_z, \qquad (5.79)$$

and the spatially dependent permittivity kernel $G_\gamma(|x - x'|)$, with the Weyl approximation [23, 25, 47, 62] as

$$G_\gamma(|x - x'|) = \frac{j}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{w_\gamma} e^{j(k_y(y - y') + k_z(z - z'))} e^{-jw_\gamma |x - x'|} dk_y dk_z, \qquad (5.80)$$

where $w_\gamma = \sqrt{(\gamma^2 - k_y^2 - k_z^2)}$. Also, E^* is just a notation denoting the two-dimensional Fourier transform in the infinite coordinates.

This simplification is necessary because the wave equation is an integro-differential equation and we need to get rid of the integration on the right hand side of the wave equation, which will give a differential equation in x. Using the vector identity $\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$, we have

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}(x, y, z; \omega)) - \nabla^2 \vec{E}(x, y, z; \omega) = \omega^2 \mu_0 \epsilon_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dy' dz' \int_0^d dx'^{\leftarrow}$$
$$\epsilon_p(|x - x'|; \omega) \cdot \vec{E}(x, y, z; \omega). \qquad (5.81)$$

We can then apply the above mentioned two-dimensional Fourier transform on $\vec{E}$ and the

Weyl approximation formula for the kernel G_γ of the tensor element ϵ_{xx} we have the three components of the wave equation as

$$\hat{x} \Rightarrow \frac{d}{dx}(\frac{d}{dx}E_x^*(x, k_y, k_z; \omega) + jk_y E_y^*(x, k_y, k_z; \omega) + jk_z E_z^*(x, k_y, k_z; \omega))$$

$$-(\frac{d^2}{d^2x} - k_y^2 - k_z^2)E_x^*(x, k_y, k_z; \omega) = \omega^2 \mu_0 \epsilon_0 \int_0^d \delta(x - x')E_x^*(x', k_y, k_z; \omega)dx'$$

$$+\frac{j\chi k_0^2}{4\pi^2}\int_0^d \frac{e^{-jw_\gamma|x-x'|}}{w_\gamma}E_x^*(x', k_y, k_z; \omega)dx', \qquad (5.82)$$

$$\hat{y} \Rightarrow jk_y(\frac{d}{dx}E_x^*(x, k_y, k_z; \omega) + jk_y E_y^*(x, k_y, k_z; \omega) + jk_z E_z^*(x, k_y, k_z; \omega))$$

$$-(\frac{d^2}{d^2x} - k_y^2 - k_z^2)E_y^*(x, k_y, k_z; \omega) = k_0^2 \epsilon_{yy}E_y^*(x, k_y, k_z; \omega), \qquad (5.83)$$

$$\hat{z} \Rightarrow jk_z(\frac{d}{dx}E_x^*(x, k_y, k_z; \omega) + jk_y E_y^*(x, k_y, k_z; \omega) + jk_z E_z^*(x, k_y, k_z; \omega))$$

$$-(\frac{d^2}{d^2x} - k_y^2 - k_z^2)E_z^*(x, k_y, k_z; \omega) = k_0^2 \epsilon_{yy}E_z^*(x, k_y, k_z; \omega), \qquad (5.84)$$

where the spatial and temporal dependence has been explicitly written for ease of understanding and the partial derivatives in x are now total derivatives, since the other spatial and temporal dependences have been taken care of through the Fourier transforms. Also, the triple integral on the right hand side of y and z components has not been written since the permittivity elements have no x dependence. The first integral on the right hand side of the x component is just k_0^2, but the second function has an x dependence in the kernel. We need to operate upon the kernel in such a way that gives us a delta function. This can be done if we multiply $\frac{e^{-jw_\gamma|x-x'|}}{w_\gamma}$ by $\frac{d^2}{d^2x} + w_\gamma^2$ [47]. This gives

$$(\frac{d^2}{d^2x} + w_\gamma^2)(\frac{e^{-jw_\gamma|x-x'|}}{w_\gamma}) = 2\pi j\delta(x - x'). \qquad (5.85)$$

Using this operation in the x component of the wave equation, we then get rid of the integral to get

$$(\frac{d^2}{d^2x} + w_\gamma^2)(\frac{d}{dx}(\frac{d}{dx}E_x^*(x, k_y, k_z; \omega) + jk_y E_y^*(x, k_y, k_z; \omega) + jk_z E_z^*(x, k_y, k_z; \omega))$$

$$-(\frac{d^2}{d^2x} - k_y^2 - k_z^2 + k_0^2)E_x^*(x, k_y, k_z; \omega)) = -\frac{\chi k_0^2}{2\pi}E_x^*(x, k_y, k_z; \omega). \qquad (5.86)$$

The above derivation in a sense means that the following field distribution function has been taken as $\vec{E}(x, y, z; \omega) = \vec{E}^*(x)e^{j(k_y y + k_z z - \omega t)}$. In order to reduce the above differential equations to an algebraic equation, therefore, requires some assumption on the x part of the distribution. Since we are looking for surface wave modes, we already have a distribution which is of the form $\vec{E}^*(x) = \vec{E}_0 e^{-\alpha x}$ [32]. Substituting this form back into the wave equations, we would get the desired algebraic form for a surface wave. So, the wave equations can be written in algebraic form as

$$\hat{x} \Rightarrow (\alpha^2 + w_\gamma^2)(-\alpha(-\alpha E_{0x} + jk_y E_{0y} + jk_z E_{0z}) - (\alpha^2 - k_y^2 - k_z^2$$
$$+k_0^2)E_{0x}) = -\frac{\chi k_0^2}{2\pi}E_{0x}, \qquad (5.87)$$

$$\hat{y} \Rightarrow jk_y(-\alpha E_{0x} + jk_y E_{0y} + jk_z E_{0z}) - (\alpha^2 - k_y^2 - k_z^2)E_{0y} = k_0^2 \epsilon_{yy} E_{0y}, \qquad (5.88)$$

$$\hat{z} \Rightarrow jk_z(-\alpha E_{0x} + jk_y E_{0y} + jk_z E_{0z}) - (\alpha^2 - k_y^2 - k_z^2)E_{0z} = k_0^2 \epsilon_{yy} E_{0z}, \qquad (5.89)$$

and we need to find the constants $\vec{E}_0$. Having gotten these equations, now we can assume propagation in y direction only ($k_z = 0$), which further simplifies the equations to

$$(\alpha^2 + w_\gamma^2)(-\alpha(-\alpha E_{0x} + jk_y E_{0y}) - (\alpha^2 - k_y^2 + k_0^2)E_{0x}) = -\frac{\chi k_0^2}{2\pi}E_{0x}, \qquad (5.90)$$

$$jk_y(-\alpha E_{0x} + jk_y E_{0y}) - (\alpha^2 - k_y^2)E_{0y} = k_0^2 \epsilon_{yy} E_{0y}, \qquad (5.91)$$

$$-(\alpha^2 - k_y^2)E_{0z} = k_0^2 \epsilon_{yy} E_{0z}, \qquad (5.92)$$

which again because of the form of the permittivity tensor has a lot of arbitrariness [49]. Therefore, we resort to only TM_y and TE_y solutions. Since we have the wave equation in $\vec{E}$, firstly, TM_y mode will be solved first and then the TE_y case through a similar wave equation in $\vec{H}$.

5.1.1 Dispersion Relation for TM_y Case with Spatial Dispersion Effects

For this case, although one needs to solve for H_z, but since the spatial dispersion is only encountered in ϵ_{xx} or the x component, E_x and E_y will be solved for and from that, we will get H_z. This technique, though strange, has been employed by Halevi [32]. This

other route is undertaken, because for a spatially dispersive media, this technique is less complex than the other one. In other words, the other way would give an integro-differential equation again, which is avoided if one uses this route. Also, $E_z = 0$, since we are dealing with TM_y case.

We have the x and y components of $\vec{E}$ field as

$$(\alpha^2 + w_\gamma^2)(-\alpha(-\alpha E_{0x} + jk_y E_{0y}) - (\alpha^2 - k_y^2 + k_0^2)E_{0x}) = -\frac{\chi k_0^2}{2\pi}E_{0x}, \qquad (5.93)$$

$$jk_y(-\alpha E_{0x} + jk_y E_{0y}) - (\alpha^2 - k_y^2)E_{0y} = k_0^2 \epsilon_{yy} E_{0y}, \qquad (5.94)$$

which can be simplified into

$$\left((\alpha^2 + w_\gamma^2)(k_y^2 - k_0^2) + \frac{\chi k_0^2}{2\pi}\right)E_{0x} - jk_y\alpha(\alpha^2 + w_\gamma^2)E_{0y} = 0, \qquad (5.95)$$

$$-jk_y\alpha E_{0x} + (-\alpha^2 - k_0^2\epsilon_{yy})E_{0y} = 0. \qquad (5.96)$$

In simple terms eqs. (5.95) and (5.96) can be written as

$$A(\alpha)E_{0x} + B(\alpha)E_{0y} = 0, \qquad (5.97)$$

$$C(\alpha)E_{0x} + D(\alpha)E_{0y} = 0, \qquad (5.98)$$

where

$$A(\alpha) = (\alpha^2 + w_\gamma^2)(k_y^2 - k_0^2) + \frac{\chi k_0^2}{2\pi}, \qquad (5.99)$$

$$B(\alpha) = -jk_y\alpha(\alpha^2 + w_\gamma^2), \qquad (5.100)$$

$$C(\alpha) = -jk_y\alpha, \qquad (5.101)$$

$$D(\alpha) = -\alpha^2 - k_0^2\epsilon_{yy}. \qquad (5.102)$$

Since A, B, C, and D are functions of α only, the secular determinant of the two equations would give us the possible roots, i.e.,

$$\Delta = \begin{vmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{vmatrix} = 0. \qquad (5.103)$$

The form of the above equations is again the same as for the general case derived in Chapter 2. Therefore, a similar procedure is used as for the general case [12]. Then, we can have either of these as the solution to the above set of linear algebraic equations

$$E_{0x} = \sum_i F_i D(\alpha_i), \qquad (5.104)$$

$$E_{0y} = -\sum_i F_i C(\alpha_i), \qquad (5.105)$$

or

$$E_{0x} = \sum_i F_i B(\alpha_i), \qquad (5.106)$$

$$E_{0y} = -\sum_i F_i A(\alpha_i), \qquad (5.107)$$

where the index i would depend on the number of roots α derived from the secular determinant of the matrix Δ. The full solution for E_{x1} and E_{y1}, the field solutions inside the semiconductor substrate, going with eqs. (5.106) and (5.107) as the solutions, are given as

$$E_{x1}(x, y; \omega) = \sum_i F_i B(\alpha_i) e^{-\alpha_i x} e^{j(k_y y - \omega t)}, \qquad (5.108)$$

$$E_{y1}(x, y; \omega) = -\sum_i F_i A(\alpha_i) e^{-\alpha_i x} e^{j(k_y y - \omega t)}. \qquad (5.109)$$

The remaining component, H_{z1}, is found from the z component of the first Maxwell's curl equation $\vec{\nabla} \times \vec{E} = j\omega\mu_0 \vec{H}$ as

$$-\alpha E_{0y} - jk_y E_{0x} = j\omega\mu_0 H_{0z}, \qquad (5.110)$$

or simply

$$H_{0z} = \frac{1}{j\omega\mu_0}(-\alpha E_{0y} - jk_y E_{0x}), \tag{5.111}$$

which has an α dependence, even in the multiplier of H_{0z}. So, the form of H_{z1} is

$$H_{z1}(x, y, z; \omega) = \frac{1}{j\omega\mu_0} \sum_i (\alpha_i F_i A(\alpha_i) - jk_y F_i B(\alpha_i)) e^{-\alpha_i x} e^{-j(k_y y - \omega t)}, \tag{5.112}$$

after substituting the respective electric field solutions found earlier. We have solved the field values inside the substrate for the TM_y case, since we have found H_{z1}, E_{x1}, and E_{y1}. The fields in free space would still remain of the same form, since it is a nondispersive medium, both temporally and spatially.

We need a field equation for H_{z2} only, since we are dealing with the TM_y mode. The form is

$$H_{z2}(x, y; \omega) = H_{1z} e^{-\beta x} e^{j(k_y y - \omega t)}, \tag{5.113}$$

where β is the positive root of the equation $\beta = \sqrt{(k_y^2 - k_0^2 \epsilon_r)}$ [2].

To match the fields across the interface, we need the tangential electric field as well. From the y component of Maxwell's curl equation, viz. $\vec{\nabla} \times \vec{H}_2 = -j\omega\epsilon_0\epsilon_r \vec{E}_2$, we get

$$\beta H_{1z} = -j\omega\epsilon_0\epsilon_r E_{1y}, \tag{5.114}$$

$$\Rightarrow E_{1y} = -\frac{\beta}{j\omega\epsilon_0\epsilon_r} H_{1z}, \tag{5.115}$$

$$\Rightarrow E_{y2}(x, y; \omega) = -\frac{\beta}{j\omega\epsilon_0\epsilon_r} H_{1z} e^{-\beta x} e^{j(k_y y - \omega t)}. \tag{5.116}$$

At present, we have four unknowns inside the substrate and one in free space, viz. F_1, F_2, F_3, F_4, and H_{1z} and have four boundary conditions, two on tangential electric field, one on the normal component of electric field, and one the tangential magnetic field [52]. This forces the reduction of the number of unknowns. Since $\alpha_1 = -\alpha_2$ and $\alpha_3 = -\alpha_4$, we can

combine the field solution in the substrate to be

$$
\begin{aligned}
E_{x1}(x,y;\omega) &= [F_1(B(\alpha_1)e^{-\alpha_1 x} + B(-\alpha_1)e^{\alpha_1 x}) \\
&\quad + F_3(B(\alpha_3)e^{-\alpha_3 x} + B(-\alpha_3)e^{\alpha_3 x})]e^{j(k_y y - \omega t)}, \quad (5.117)
\end{aligned}
$$

$$
\begin{aligned}
E_{y1}(x,y;\omega) &= -[F_1(A(\alpha_1)e^{-\alpha_1 x} + A(-\alpha_1)e^{\alpha_1 x}) \\
&\quad + F_3(A(\alpha_3)e^{-\alpha_3 x} + A(-\alpha_3)e^{\alpha_3 x})]e^{j(k_y y - \omega t)}, \quad (5.118)
\end{aligned}
$$

$$
\begin{aligned}
H_{z1}(x,y;\omega) &= \frac{1}{j\omega\mu_0}[F_1([\alpha_1 A(\alpha_1) - jk_y B(\alpha_1)]e^{-\alpha_1 x} + [-\alpha_1 A(-\alpha_1) \\
&\quad - jk_y B(-\alpha_1)]e^{\alpha_1 x}) + F_3([\alpha_3 A(\alpha_3) - jk_y B(\alpha_3)]e^{-\alpha_3 x} \\
&\quad - [\alpha_3 A(-\alpha_3) + jk_y B(-\alpha_3)]e^{\alpha_3 x})]e^{-j(k_y y - \omega t)}. \quad (5.119)
\end{aligned}
$$

We have a fourth boundary condition on the normal component of electric field, D_x. The continuity of this boundary condition depends on the thickness of the depletion charge region formed at the interface [59]. This thickness depends basically on the inter-lattice separation and the frequency of operation. But because medium 1 is spatially dispersive and moreover, ϵ_{xx} is spatially dependent, matching will cause a problem. Therefore, only the tangential fields will be matched at $x = d$.

Now that the number of unknowns is reduced to three in count, we have have sufficient boundary conditions to get a unique dispersion relation [52]. The tangential boundary condition at $x = 0$, i.e., $E_{y1}|_{x=0} = 0$ gives

$$
F_1(A(\alpha_1) + A(-\alpha_1)) + F_3(A(\alpha_3) + A(-\alpha_3)) = 0, \quad (5.120)
$$

and at the interface $x = d$ gives

$$
\begin{aligned}
F_1(A(\alpha_1)e^{-\alpha_1 d} + A(-\alpha_1)e^{\alpha_1 d}) &+ F_3(A(\alpha_3)e^{-\alpha_3 d} + A(-\alpha_3)e^{\alpha_3 d}) \\
&= \frac{\beta}{j\omega\epsilon_0\epsilon_r}H_{1z}e^{-\beta d}. \quad (5.121)
\end{aligned}
$$

The continuity of tangential magnetic field at $x = d$ finally leads to

$$\frac{1}{j\omega\mu_0}[F_1([\alpha_1 A(\alpha_1) - jk_y B(\alpha_1)]e^{-\alpha_1 d} + [-\alpha_1 A(-\alpha_1) - jk_y B(-\alpha_1)]e^{\alpha_1 d})$$
$$+F_3([\alpha_3 A(\alpha_3) - jk_y B(\alpha_3)]e^{-\alpha_3 d} + [-\alpha_3 A(-\alpha_3) - jk_y B(-\alpha_3)]e^{\alpha_3 d})] = H_{1z}e^{-\beta d}.$$

$$(5.122)$$

We have solved for the dispersion relation and in matrix form, the relation is written as

$$\begin{bmatrix} A(\alpha_1) + A(-\alpha_1) & A(\alpha_3) + A(-\alpha_3) & 0 \\ M_1 & M_3 & \frac{\beta}{j\omega\epsilon_0\epsilon_r}e^{-\beta d} \\ N_1 & N_3 & -e^{-\beta d} \end{bmatrix} \begin{bmatrix} F_1 \\ F_3 \\ H_{1z} \end{bmatrix} = 0, \qquad (5.123)$$

where

$$M_1 = A(\alpha_1)e^{-\alpha_1 d} + A(-\alpha_1)e^{\alpha_1 d}, \qquad (5.124)$$

$$M_3 = A(\alpha_3)e^{-\alpha_3 d} + A(-\alpha_3)e^{\alpha_3 d}, \qquad (5.125)$$

and

$$N_1 = \frac{1}{j\omega\mu_0}\left([\alpha_1 A(\alpha_1) - jk_y B(\alpha_1)]e^{-\alpha_1 d} + [-\alpha_1 A(-\alpha_1) - jk_y B(-\alpha_1)]e^{\alpha_1 d}\right), \quad (5.126)$$

$$N_3 = \frac{1}{j\omega\mu_0}\left([\alpha_3 A(\alpha_3) - jk_y B(\alpha_3)]e^{-\alpha_3 d} + [-\alpha_3 A(-\alpha_3) - jk_y B(-\alpha_3)]e^{\alpha_3 d}\right). \quad (5.127)$$

The determinant of eq. (5.123) gives the required dispersion relation.

5.1.2 Results for TM_y Mode

To see the effect of spatial dispersion in absence of a steady magnetic field, the derived dispersion equation was run using GA and compared against TM_y mode when spatial dispersion was neglected. The plot of the two dispersion curves is shown in fig. 5.1.

The first difference that can be noticed is that there are some surface poles missing in the spatial dispersion and some have shifted. This difference could be due to additional
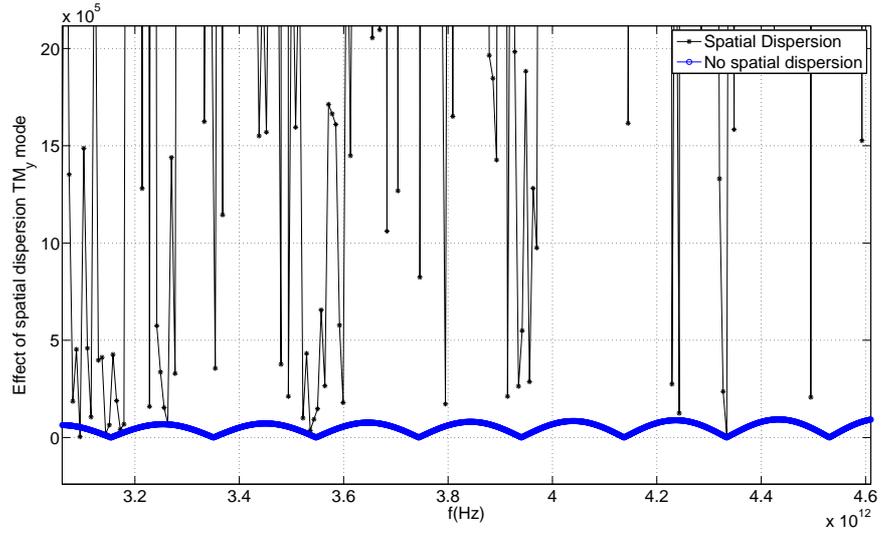
Fig. 5.1: Effect of spatial dispersion TM_y mode.

roots (α) for the spatial dispersion. When a semilogy plot for fig. 5.1 was plotted for a higher resolution for the same range, surface poles close to the spatially nondispersive case were found indicated by arrows in fig. 5.2.

5.1.3   Dispersion Relation for TE_y Case

The derivation of this case is similar to that of the TM_y case. The only difference is that instead of solving the wave equation for $\vec{E}$, we solve the wave equation in $\vec{H}$. The form of these wave equations is the same as for the electric fields [32]. Mathematically, we have

$$\hat{x} \Rightarrow (\alpha^2 + w_\gamma^2)(-\alpha(-\alpha H_{0x} + jk_y H_{0y} + jk_z H_{0z}) - (\alpha^2 - k_y^2 - k_z^2 + k_0^2)H_{0x})$$
$$= -\frac{\chi k_0^2}{2\pi}H_{0x}, \quad (5.128)$$

$$\hat{y} \Rightarrow jk_y(-\alpha H_{0x} + jk_y H_{0y} + jk_z H_{0z}) - (\alpha^2 - k_y^2 - k_z^2)H_{0y} = k_0^2 \epsilon_{yy} H_{0y}, \quad (5.129)$$

$$\hat{z} \Rightarrow jk_z(-\alpha H_{0x} + jk_y H_{0y} + jk_z H_{0z}) - (\alpha^2 - k_y^2 - k_z^2)H_{0z} = k_0^2 \epsilon_{yy} H_{0z}. \quad (5.130)$$
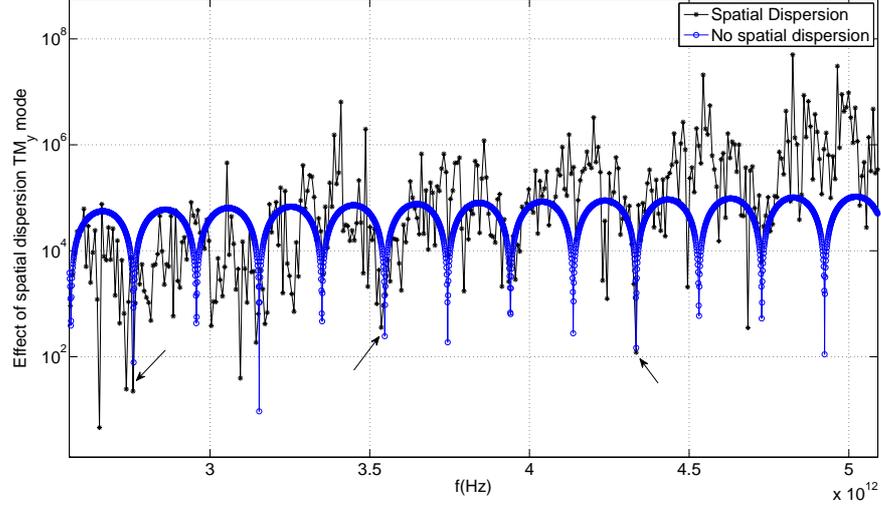
Fig. 5.2: Effect of spatial dispersion TM_y mode on semilogy plot.

Going through the same procedure as for TE_y case, we have

$$(\alpha^2 + w_\gamma^2)(-\alpha(-\alpha H_{0x} + jk_y H_{0y}) - (\alpha^2 - k_y^2 + k_0^2)H_{0x}) = -\frac{\chi k_0^2}{2\pi}H_{0x}, \qquad (5.131)$$

$$jk_y(-\alpha H_{0x} + jk_y H_{0y}) - (\alpha^2 - k_y^2)H_{0y} = k_0^2 \epsilon_{yy} H_{0y}, \qquad (5.132)$$

$$-(\alpha^2 - k_y^2)H_{0z} = k_0^2 \epsilon_{yy} H_{0z}, \qquad (5.133)$$

from which we can see that again, we have the same arbitrariness, since H_z is completely independent of the other two fields. We, therefore, are forced to have a transverse mode TM_y. This mode makes $H_z = 0$ and the field quantities we need to solve for are E_z, H_x, and H_y. Since H_x and H_y are dependent, we solve these first and using Maxwell's curl equation, we will get E_z.

So, we have

$$\left((\alpha^2 + w_\gamma^2)(k_y^2 - k_0^2) + \frac{\chi k_0^2}{2\pi}\right)H_{0x} - jk_y\alpha(\alpha^2 + w_\gamma^2)H_{0y} = 0, \qquad (5.134)$$

$$-jk_y\alpha H_{0x} + (-\alpha^2 - k_0^2\epsilon_{yy})H_{0y} = 0, \qquad (5.135)$$

or simply

$$A(\alpha)H_{0x} + B(\alpha)H_{0y} = 0, \tag{5.136}$$

$$C(\alpha)H_{0x} + D(\alpha)H_{0y} = 0, \tag{5.137}$$

where $A, B, C,$ and D have been defined in the TE_y derivation. The roots can be found by equating the determinant of the matrix

$$\begin{bmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{bmatrix}, \tag{5.138}$$

and equating it to zero, i.e.,

$$\Delta = \begin{vmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{vmatrix} = 0. \tag{5.139}$$

From the above determinant, we get i roots, so that the solutions to the coefficients H_{0x} and H_{0y} can be written as

$$H_{0x} = \sum_i F_i D(\alpha_i), \tag{5.140}$$

$$H_{0y} = -\sum_i F_i C(\alpha_i), \tag{5.141}$$

or

$$H_{0x} = \sum_i F_i B(\alpha_i), \tag{5.142}$$

$$H_{0y} = -\sum_i F_i A(\alpha_i). \tag{5.143}$$

The full solution is then given as

$$H_{x1}(x, y; \omega) = \sum_i F_i D(\alpha_i) e^{-\alpha_i x} e^{j(k_y y - \omega t)}, \tag{5.144}$$

$$H_{y1}(x, y; \omega) = -\sum_i F_i C(\alpha_i) e^{-\alpha_i x} e^{j(k_y y - \omega t)}, \tag{5.145}$$

where the first solution of the coefficients, i.e., eqs. (5.140) and (5.141), have been substituted. Then, the E_z field can be found from the z component of the curl equation on $\vec{H}$, i.e., $\hat{z} \cdot \vec{\nabla} \times \vec{H} = j\omega\epsilon_{yy}E_z$, which is written as

$$-\alpha H_{0y} - jk_y H_{0x} = j\omega\epsilon_{yy}E_{0z}, \tag{5.146}$$

$$\Rightarrow E_{0z} = \frac{1}{j\omega\epsilon_{yy}}(-\alpha H_{0y} - jk_y H_{0x}), \tag{5.147}$$

$$\Rightarrow E_{0z} = \frac{1}{j\omega\epsilon_{yy}}\sum_i (\alpha_i F_i C(\alpha_i) - jk_y F_i D(\alpha_i)). \tag{5.148}$$

The full solution to E_{z1} is then given as

$$E_{1z}(x,y;\omega) = \frac{1}{j\omega\epsilon_{yy}}\sum_i (\alpha_i F_i C(\alpha_i) - jk_y F_i D(\alpha_i))e^{-\alpha_i x}e^{j(k_y y - \omega t)}. \tag{5.149}$$

Again, we are forced to reduce the number of unknowns, since we have four boundary conditions ($\vec{E}_1|_{x=0} = 0$, continuity of $\vec{E}|_{x=d}$, continuity of tangential $\vec{H}|_{x=d}$, and continuity of normal $\vec{B}/\vec{H}$ at $x = d$) and five unknowns. Observing that $\alpha_1 = -\alpha_2$ and $\alpha_3 = -\alpha_4$, we have

$$\begin{aligned} E_{1z}(x,y;\omega) &= \frac{1}{j\omega\epsilon_{yy}}[F_1([\alpha_1 C(\alpha_1) - jk_y D(\alpha_1)]e^{-\alpha_1 x} + [-\alpha_1 C(-\alpha_1) - jk_y D(-\alpha_1)]e^{\alpha_1 x}) \\ &\quad + F_3([\alpha_3 C(\alpha_3) - jk_y D(\alpha_3)]e^{-\alpha_3 x} \\ &\quad + [-\alpha_3 C(-\alpha_3) - jk_y D(-\alpha_3)]e^{\alpha_3 x})]e^{j(k_y y - \omega t)}, \end{aligned} \tag{5.150}$$

$$\begin{aligned} H_{x1}(x,y;\omega) &= [F_1(D(\alpha_1)e^{-\alpha_1 x} + D(-\alpha_1)e^{\alpha_1 x}) + F_3(D(\alpha_3)e^{-\alpha_3 x} \\ &\quad + D(-\alpha_3)e^{\alpha_3 x})]e^{j(k_y y - \omega t)}, \end{aligned} \tag{5.151}$$

$$\begin{aligned} H_{y1}(x,y;\omega) &= -[F_1(C(\alpha_1)e^{-\alpha_1 x} + C(-\alpha_1)e^{\alpha_1 x}) + F_3(C(\alpha_3)e^{-\alpha_3 x} \\ &\quad + C(-\alpha_3)e^{\alpha_3 x})]e^{j(k_y y - \omega t)}. \end{aligned} \tag{5.152}$$

Having found the fields inside the substrate and knowing that for a TE_y mode, $E_{z2}(x,y;\omega) = E_{1z}e^{-\beta x}e^{j(k_y - \omega t)}$, is the only field in free space that is needed to find the rest of the fields [2],

we will use the y and x component of the curl equation $\vec{\nabla} \times \vec{E} = j\omega\mu_0\vec{H}$,

$$\beta E_{1z} = j\omega\mu_0 H_{1y}, \tag{5.153}$$

$$H_{1y} = \frac{\beta}{j\omega\mu_0}E_{1z}, \tag{5.154}$$

$$\Rightarrow H_{y2}(x,y;\omega) = \frac{\beta}{j\omega\mu_0}E_{1z}e^{-\beta x}e^{j(k_y y-\omega t)}, \tag{5.155}$$

and

$$jk_y E_{1z} = j\omega\mu_0 H_{1x}, \tag{5.156}$$

$$H_{1x} = \frac{k_y}{\omega\mu_0}E_{1z}, \tag{5.157}$$

$$\Rightarrow H_{x2}(x,y;\omega) = \frac{k_y}{\omega\mu_0}E_{1z}e^{-\beta x}e^{j(k_y y-\omega t)}. \tag{5.158}$$

Before applying the boundary conditions, since after reducing the number of roots from four to two, we have four boundary conditions for three unknowns, the continuity of fields at the interface $x = d$ will be used. Applying the boundary conditions at the interface, we get

$$\frac{1}{j\omega\epsilon_{yy}}[F_1([\alpha_1 C(\alpha_1) - jk_y D(\alpha_1)]e^{-\alpha_1 d} + [-\alpha_1 C(-\alpha_1) - jk_y D(-\alpha_1)]e^{\alpha_1 d})$$
$$+F_3([\alpha_3 C(\alpha_3) - jk_y D(\alpha_3)]e^{-\alpha_3 d} + [-\alpha_3 C(-\alpha_3) - jk_y D(-\alpha_3)]e^{\alpha_3 d})] = A_z(\beta),$$
$$\tag{5.159}$$

$$-[F_1(C(\alpha_1)e^{-\alpha_1 d} + C(-\alpha_1)e^{\alpha_1 d}) + F_3(C(\alpha_3)e^{-\alpha_3 d} + C(-\alpha_3)e^{\alpha_3 d})] = \frac{\beta}{j\omega\mu_0}A_z(\beta),$$
$$\tag{5.160}$$

$$F_1(D(\alpha_1)e^{-\alpha_1 d} + D(-\alpha_1)e^{\alpha_1 d}) + F_3(D(\alpha_3)e^{-\alpha_3 d} + D(-\alpha_3)e^{\alpha_3 d}) = \frac{k_y}{\omega\mu_0}A_z(\beta),$$
$$\tag{5.161}$$

where $A_z(\beta) = E_{1z}e^{-\beta d}$. Writing the derived equations in matrix form, we get

$$\begin{bmatrix} N_1 & N_3 & -e^{-\beta d} \\ C_{1spat} & C_{3spat} & \frac{\beta}{j\omega\mu_0}e^{-\beta d} \\ D_{1spat} & D_{3spat} & -\frac{k_y}{\omega\mu_0}e^{-\beta d} \end{bmatrix} \begin{bmatrix} F_1 \\ F_3 \\ E_{1z} \end{bmatrix} = 0, \tag{5.162}$$

where

$$C_{1spat} = C(\alpha_1)e^{-\alpha_1 d} + C(-\alpha_1)e^{\alpha_1 d}, \tag{5.163}$$

$$C_{3spat} = C(\alpha_3)e^{-\alpha_3 d} + C(-\alpha_3)e^{\alpha_3 d}, \tag{5.164}$$

and

$$D_{1spat} = D(\alpha_1)e^{-\alpha_1 d} + D(-\alpha_1)e^{\alpha_1 d}, \tag{5.165}$$

$$D_{3spat} = D(\alpha_3)e^{-\alpha_3 d} + D(-\alpha_3)e^{\alpha_3 d}. \tag{5.166}$$

Also,

$$N_1 = \frac{1}{j\omega\epsilon_{yy}}([\alpha_1 C(\alpha_1) - jk_y D(\alpha_1)]e^{-\alpha_1 d} \tag{5.167}$$
$$+[-\alpha_1 C(-\alpha_1) - jk_y D(-\alpha_1)]e^{\alpha_1 d}),$$

$$N_3 = \frac{1}{j\omega\epsilon_{yy}}([\alpha_3 C(\alpha_3) - jk_y D(\alpha_3)]e^{-\alpha_3 d} \tag{5.168}$$
$$+[-\alpha_3 C(-\alpha_3) - jk_y D(-\alpha_3)]e^{\alpha_3 d}).$$

The determinant of eq. (5.162) gives a dispersion relation for TE_y mode.

5.1.4   Results for TE_y Mode

Again, the same case of comparing plots of dispersion curves with and without spatial dispersion is undertaken. The plot is shown in fig. 5.3.

Compared to TM_y mode, spatial dispersion causes a lot more surface modes to appear. More analysis needs to be done on these modes to understand why the poles shift and what
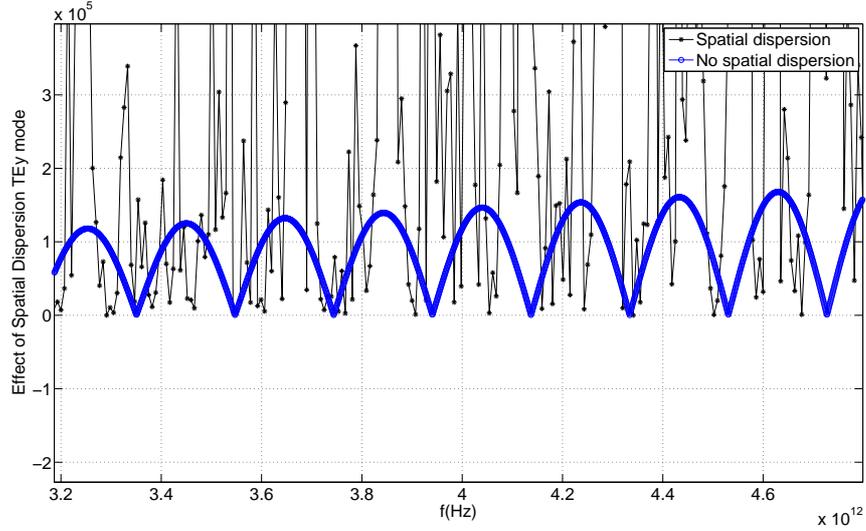
Fig. 5.3: Effect of spatial dispersion TE_y mode.

causes this shift. A high resolution case for spatial dispersion was run and similar to TM_y mode, there were surface poles in close proximity to the nonspatially dispersive case as shown in fig. 5.4. It has to be said that the results are not conclusive and more work needs to be done before something can be said about the effect of spatial dispersion.

5.2 $\tilde{\epsilon}_p$ with $\vec{B}_0 \neq 0$

The derivation of $\tilde{\epsilon}_p$ will be carried out for the most general orientation of the steady magnetic field, i.e., $\vec{B}_0 = \hat{x}cos\theta + \hat{y}sin\theta$ [64, 65]. Then, as done before, simplifying assumptions will be made to get to a dispersion relation. With a finite magnetic field, along with the pressure term and finite damping, the first order perturbations in the fluid equations for holes and electrons can be written as

$$-j\omega\vec{u}_h = \frac{q}{m_h}(\vec{E} + \vec{u}_h \times \vec{B}_0) - \frac{\vec{\nabla}p_h}{m_h N_0} + \nu_{he}(\vec{u}_e - \vec{u}_h), \qquad (5.169)$$

$$-j\omega\vec{u}_e = -\frac{q}{m_e}(\vec{E} + \vec{u}_e \times \vec{B}_0) - \frac{\vec{\nabla}p_e}{m_e N_0} + \nu_{eh}(\vec{u}_h - \vec{u}_e), \qquad (5.170)$$

$$-j\omega n_h + N_0\vec{\nabla} \cdot \vec{u}_h = \frac{n_h}{t_h}, \qquad (5.171)$$

$$-j\omega n_e + N_0\vec{\nabla} \cdot \vec{u}_e = \frac{n_e}{t_e}, \qquad (5.172)$$
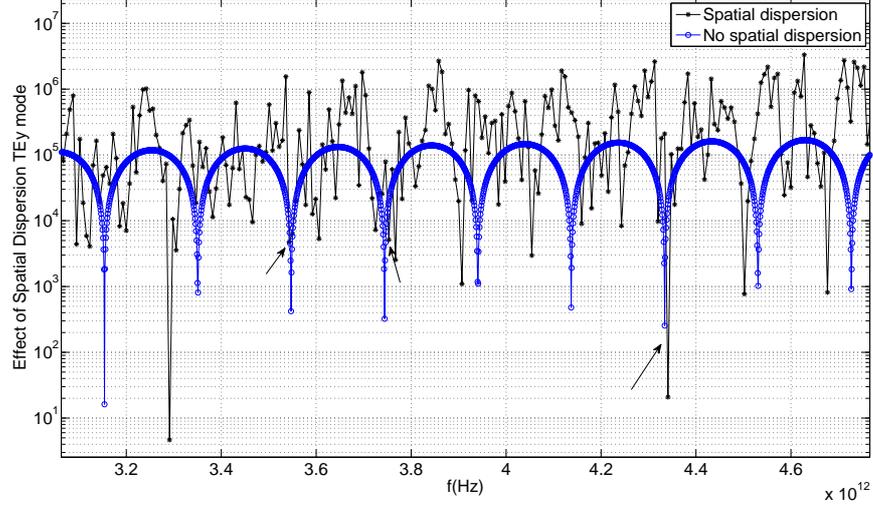
Fig. 5.4: Effect of spatial dispersion TE_y mode semilogy plot.

and the recombination rates are finite so that the continuity equations are inhomogeneous or in other words, the $P - L$ term is nonzero [59, 60].

Using the degenerate gas model for holes and electrons as used by Halevi et al. [32], the pressure terms can be substituted for in terms of density perturbations and thereafter by fluid velocities using the continuity equation. Working along these lines, firstly, using a degenerate gas model, eqs. (5.169) and (5.170) can be modified as

$$\vec{u}_h = \frac{jq}{\omega m_h}(\vec{E} + \vec{u}_h \times \vec{B}_0) - \frac{jv_{fh}^2}{3\omega N_0}\vec{\nabla}n_h + \frac{j\nu_{he}}{\omega}(\vec{u}_e - \vec{u}_h), \tag{5.173}$$

$$\vec{u}_e = -\frac{jq}{\omega m_e}(\vec{E} + \vec{u}_e \times \vec{B}_0) - \frac{jv_{fe}^2}{3\omega N_0}\vec{\nabla}n_e + \frac{j\nu_{eh}}{\omega}(\vec{u}_h - \vec{u}_e). \tag{5.174}$$

The continuity equations for holes and electrons, i.e.,

$$n_h = \frac{N_0}{j\omega + \frac{1}{t_h}}\vec{\nabla} \cdot \vec{u}_h, \tag{5.175}$$

$$n_e = \frac{N_0}{j\omega + \frac{1}{t_e}}\vec{\nabla} \cdot \vec{u}_e, \tag{5.176}$$

are then used to substitute for n_s in the momentum equation. Thus, eqs. (5.173) and (5.174) can be rewritten as

$$\vec{u}_h = \frac{jq}{\omega m_h}(\vec{E} + \vec{u}_h \times \vec{B}_0) - \frac{j\beta_h^2}{\omega(j\omega + \frac{1}{t_h})}\vec{\nabla}(\vec{\nabla} \cdot \vec{u}_h) + \frac{j\nu_{he}}{\omega}(\vec{u}_e - \vec{u}_h), \qquad (5.177)$$

$$\vec{u}_e = -\frac{jq}{\omega m_e}(\vec{E} + \vec{u}_e \times \vec{B}_0) - \frac{j\beta_e^2}{\omega(j\omega + \frac{1}{t_e})}\vec{\nabla}(\vec{\nabla} \cdot \vec{u}_e) + \frac{j\nu_{eh}}{\omega}(\vec{u}_h - \vec{u}_e), \qquad (5.178)$$

where $\beta_s^2 = \frac{v_{fs}^2}{3}$. Now that we have a pair of equations that have $\vec{u}_s$ and $\vec{E}$ as the only variables, the cross product can be performed to get a matrix representation of the equations. This is done as (with s representing either of the species),

$$\vec{u}_s \times \vec{B}_0 = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_{sx} & u_{sy} & u_{sz} \\ B_0 cos\theta & B_0 sin\theta & 0 \end{bmatrix}, \qquad (5.179)$$

$$\Rightarrow \vec{u}_s \times \vec{B}_0 = -\hat{x}(u_{sz}B_0 sin\theta) + \hat{y}(u_{sz}B_0 cos\theta) + \hat{z}B_0(u_{sx}sin\theta - u_{sy}cos\theta). \quad (5.180)$$

The momentum equations can then be written as [65]

$$\vec{u}_h = \frac{jq}{\omega m_h}(\vec{E} - \hat{x}(u_{hz}B_0 sin\theta) + \hat{y}(u_{hz}B_0 cos\theta) + \hat{z}B_0(u_{hx}sin\theta - u_{hy}cos\theta)) \qquad (5.181)$$

$$-\frac{j\beta_h^2}{\omega(j\omega + \frac{1}{t_h})}\vec{\nabla}(\vec{\nabla} \cdot \vec{u}_h) + \frac{j\nu_{he}}{\omega}(\vec{u}_e - \vec{u}_h),$$

$$\vec{u}_e = -\frac{jq}{\omega m_e}(\vec{E} - \hat{x}(u_{ez}B_0 sin\theta) + \hat{y}(u_{ez}B_0 cos\theta) + \hat{z}B_0(u_{ex}sin\theta - u_{ey}cos\theta)) \qquad (5.182)$$

$$-\frac{j\beta_e^2}{\omega(j\omega + \frac{1}{t_e})}\vec{\nabla}(\vec{\nabla} \cdot \vec{u}_e) + \frac{j\nu_{eh}}{\omega}(\vec{u}_h - \vec{u}_e).$$

With the same assumption as before, i.e., $\frac{\partial}{\partial y} = 0$ and $\frac{\partial}{\partial z} = 0$ in the $\vec{\nabla}(\vec{\nabla} \cdot \vec{u}_s)$ term or no density variation in directions parallel to the interface, the above vector differential equation

is reduced to an algebraic equation,

$$\vec{u}_h = \frac{jq}{\omega m_h}(\vec{E} - \hat{x}(u_{hz}B_0 sin\theta) + \hat{y}(u_{hz}B_0 cos\theta) + \hat{z}B_0(u_{hx}sin\theta - u_{hy}cos\theta)) \quad (5.183)$$

$$-\frac{j\beta_h^2}{\omega(j\omega + \frac{1}{t_h})}\hat{x}jk_x(jk_x u_{hx}) + \frac{j\nu_{he}}{\omega}(\vec{u}_e - \vec{u}_h),$$

$$\vec{u}_e = -\frac{jq}{\omega m_e}(\vec{E} - \hat{x}(u_{ez}B_0 sin\theta) + \hat{y}(u_{ez}B_0 cos\theta) + \hat{z}B_0(u_{ex}sin\theta - u_{ey}cos\theta)) \quad (5.184)$$

$$-\frac{j\beta_e^2}{\omega(j\omega + \frac{1}{t_e})}\hat{x}jk_x(jk_x u_{ex}) + \frac{j\nu_{eh}}{\omega}(\vec{u}_h - \vec{u}_e),$$

the components of which are

$$\hat{x} \Rightarrow u_{hx} + \frac{j\Omega_{ch}}{\omega}sin\theta u_{hz} = \frac{jq}{\omega m_h}E_x + \frac{j\beta_h^2}{\omega(j\omega + \frac{1}{t_h})}(k_x^2 u_{hx}) + \frac{j\nu_{he}}{\omega}(u_{ex} - u_{hx}), \quad (5.185)$$

$$\hat{y} \Rightarrow u_{hy} - \frac{j\Omega_{ch}}{\omega}cos\theta u_{hz} = \frac{jq}{\omega m_h}E_y + \frac{j\nu_{he}}{\omega}(u_{ey} - u_{hy}), \quad (5.186)$$

$$\hat{z} \Rightarrow u_{hz} - \frac{j\Omega_{ch}}{\omega}sin\theta u_{hx} + \frac{j\Omega_{ch}}{\omega}cos\theta u_{hy} = \frac{jq}{\omega m_h}E_z + \frac{j\nu_{he}}{\omega}(u_{ez} - u_{hz}), \quad (5.187)$$

and

$$\hat{x} \Rightarrow u_{ex} - \frac{j\Omega_{ce}}{\omega}sin\theta u_{ez} = -\frac{jq}{\omega m_e}E_x + \frac{j\beta_e^2}{\omega(j\omega + \frac{1}{t_e})}(k_x^2 u_{ex}) + \frac{j\nu_{eh}}{\omega}(u_{hx} - u_{ex}), \quad (5.188)$$

$$\hat{y} \Rightarrow u_{ey} + \frac{j\Omega_{ce}}{\omega}cos\theta u_{ez} = -\frac{jq}{\omega m_e}E_y + \frac{j\nu_{eh}}{\omega}(u_{hy} - u_{ey}), \quad (5.189)$$

$$\hat{z} \Rightarrow u_{ez} + \frac{j\Omega_{ce}}{\omega}sin\theta u_{ex} - \frac{j\Omega_{ce}}{\omega}cos\theta u_{ey} = -\frac{jq}{\omega m_e}E_z + \frac{j\nu_{eh}}{\omega}(u_{hz} - u_{ez}), \quad (5.190)$$

for holes and electrons, respectively. Before writing the above equations in matrix form, we assume infinite recombination time, i.e., $t_s = \infty$, implying no production or loss terms in the continuity equation. Also, with the effective masses of holes and electrons being comparable [60], we have $\beta_e \simeq \beta_h \simeq \beta_T$. Then in matrix form, the above set of six

equations can be written as

$$
\begin{bmatrix}
1 - \frac{\beta_T^2}{\omega^2}k_x^2 & 0 & \frac{j\Omega_{ch}}{\omega}sin\theta \\
0 & 1 & -\frac{j\Omega_{ch}}{\omega}cos\theta \\
-\frac{j\Omega_{ch}}{\omega}sin\theta & \frac{j\Omega_{ch}}{\omega}cos\theta & 1
\end{bmatrix}
\begin{bmatrix}
u_{hx} \\
u_{hy} \\
u_{hz}
\end{bmatrix}
=
\frac{jq}{\omega m_h}\vec{E} + \frac{j\nu_{he}}{\omega}(\vec{u}_e - \vec{u}_h), \quad (5.191)
$$

$$
\begin{bmatrix}
1 - \frac{\beta_T^2}{\omega^2}k_x^2 & 0 & -\frac{j\Omega_{ce}}{\omega}sin\theta \\
0 & 1 & \frac{j\Omega_{ce}}{\omega}cos\theta \\
\frac{j\Omega_{ce}}{\omega}sin\theta & -\frac{j\Omega_{ce}}{\omega}cos\theta & 1
\end{bmatrix}
\begin{bmatrix}
u_{ex} \\
u_{ey} \\
u_{ez}
\end{bmatrix}
=
-\frac{jq}{\omega m_e}\vec{E} + \frac{j\nu_{eh}}{\omega}(\vec{u}_h - \vec{u}_e). \quad (5.192)
$$

In compact form, eqs. (5.191) and (5.192) are written as

$$
M_h \cdot \vec{u}_h = \frac{jq}{\omega m_h}\vec{E} + \frac{j\nu_{he}}{\omega}(\vec{u}_e - \vec{u}_h), \tag{5.193}
$$

$$
M_e \cdot \vec{u}_e = -\frac{jq}{\omega m_e}\vec{E} + \frac{j\nu_{eh}}{\omega}(\vec{u}_h - \vec{u}_e). \tag{5.194}
$$

Since the two matrices, M_h and M_e, are not the same, before subtracting eqs. (5.193) and (5.194) we need to multiply them by the respective matrix inverse, i.e., M_h^{-1} and M_e^{-1}. Doing so, we get

$$
\vec{u}_h = \frac{jq}{\omega m_h}M_h^{-1} \cdot \vec{E} + \frac{j\nu_{he}}{\omega}M_h^{-1} \cdot (\vec{u}_e - \vec{u}_h), \tag{5.195}
$$

$$
\vec{u}_e = -\frac{jq}{\omega m_e}M_e^{-1} \cdot \vec{E} + \frac{j\nu_{eh}}{\omega}M_e^{-1} \cdot (\vec{u}_h - \vec{u}_e), \tag{5.196}
$$

and after subtracting to get

$$
[I + \frac{j\nu_{he}}{\omega}M_h^{-1} + \frac{j\nu_{eh}}{\omega}M_e^{-1}] \cdot (\vec{u}_h - \vec{u}_e) = (\frac{jq}{\omega m_h}M_h^{-1} + \frac{jq}{\omega m_e}M_e^{-1}) \cdot \vec{E}. \tag{5.197}
$$

Let $A = I + \frac{j\nu_{he}}{\omega}M_h^{-1} + \frac{j\nu_{eh}}{\omega}M_e^{-1}$. Then, multiplying the above equation with A^{-1} we get an equation that relates the fluid velocity difference to the electric field (both perturbed quantities)

$$
(\vec{u}_h - \vec{u}_e) = (\frac{jq}{\omega m_h}A^{-1}M_h^{-1} + \frac{jq}{\omega m_e}A^{-1}M_e^{-1}) \cdot \vec{E}. \tag{5.198}
$$

Knowing that the current density is given as $\vec{J} = N_0 q(\vec{u}_h - \vec{u}_e)$, this expression is substituted into the $\vec{\nabla} \times \vec{H}$ equation, to get a tensor that combines the effect of a static magnetic field and spatial dispersion, as has also been done before,

$$\vec{J} = (\frac{jN_0q^2}{\omega m_h}A^{-1}M_h^{-1} + \frac{jN_0q^2}{\omega m_e}A^{-1}M_e^{-1}) \cdot \vec{E}, \tag{5.199}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} - j\omega\epsilon_0\vec{E}, \tag{5.200}$$

$$\Rightarrow \vec{\nabla} \times \vec{H} = [\frac{jN_0q^2}{\omega m_h}A^{-1}M_h^{-1} + \frac{jN_0q^2}{\omega m_e}A^{-1}M_e^{-1} - j\omega\epsilon_0 I] \cdot \vec{E}, \tag{5.201}$$

$$\Rightarrow \vec{\nabla} \times \vec{H} = -j\omega\epsilon_0[I - (\frac{\omega_{ph}^2}{\omega^2}A^{-1}M_h^{-1} + \frac{\omega_{pe}^2}{\omega^2}A^{-1}M_e^{-1})] \cdot \vec{E}, \tag{5.202}$$

$$\Rightarrow \tilde{\epsilon}_p(k_x; \omega) = I - (\frac{\omega_{ph}^2}{\omega^2}A^{-1}M_h^{-1} + \frac{\omega_{pe}^2}{\omega^2}A^{-1}M_e^{-1}). \tag{5.203}$$

Having derived $\tilde{\epsilon}_p$, we now can consider different configurations of $\vec{B}_0$, to simplify the algebra [12, 29, 49, 50]. As done before, $\vec{B}_0$ perpendicular and parallel to the interface will be analyzed.

5.3 Dispersion Relation with $\vec{B}_0$ Perpendicular to the Interface

This case is derived by substituting $\theta = 0°$ in the above matrices [50]. Thus, M_e and M_h become

$$M_e = \begin{bmatrix} 1 - \frac{\beta_T^2}{\omega^2}k_x^2 & 0 & 0 \\ 0 & 1 & jX_e \\ 0 & -jX_e & 1 \end{bmatrix}, \tag{5.204}$$

$$M_h = \begin{bmatrix} 1 - \frac{\beta_T^2}{\omega^2}k_x^2 & 0 & 0 \\ 0 & 1 & -jX_h \\ 0 & jX_h & 1 \end{bmatrix}, \tag{5.205}$$

where $X_s = \frac{\Omega_{cs}}{\omega}$. We need A^{-1}, M_e^{-1}, and M_h^{-1} to get to a final expression for $\tilde{\epsilon}_p$. M_e^{-1} and M_h^{-1} are

$$
M_e^{-1} = \begin{bmatrix} \frac{1}{1-\frac{\beta_T^2}{\omega^2}k_x^2} & 0 & 0 \\ 0 & \frac{1}{1-X_e^2} & \frac{-jX_e}{1-X_e^2} \\ 0 & \frac{jX_e}{1-X_e^2} & \frac{1}{1-X_e^2} \end{bmatrix}, \tag{5.206}
$$

$$
M_h^{-1} = \begin{bmatrix} \frac{1}{1-\frac{\beta_T^2}{\omega^2}k_x^2} & 0 & 0 \\ 0 & \frac{1}{1-X_h^2} & \frac{jX_h}{1-X_h^2} \\ 0 & \frac{-jX_h}{1-X_h^2} & \frac{1}{1-X_h^2} \end{bmatrix}. \tag{5.207}
$$

Then, $A = I + \frac{j\nu_{eh}}{\omega}M_e^{-1} + \frac{j\nu_{he}}{\omega}M_h^{-1}$, with $\frac{\nu_{eh}}{\omega} = \Gamma_e$ and $\frac{\nu_{he}}{\omega} = \Gamma_h$, is given as

$$
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{j\Gamma_e}{1-\frac{\beta_T^2}{\omega^2}k_x^2} & 0 & 0 \\ 0 & \frac{j\Gamma_e}{1-X_e^2} & \frac{\Gamma_e X_e}{1-X_e^2} \\ 0 & \frac{-\Gamma_e X_e}{1-X_e^2} & \frac{j\Gamma_e}{1-X_e^2} \end{bmatrix} + \begin{bmatrix} \frac{j\Gamma_h}{1-\frac{\beta_T^2}{\omega^2}k_x^2} & 0 & 0 \\ 0 & \frac{j\Gamma_h}{1-X_h^2} & \frac{-\Gamma_h X_h}{1-X_h^2} \\ 0 & \frac{\Gamma_h X_h}{1-X_h^2} & \frac{j\Gamma_h}{1-X_h^2} \end{bmatrix}, \tag{5.208}
$$

which after adding the matrices, gives

$$
A = \begin{bmatrix} 1 + \frac{j(\Gamma_e+\Gamma_h)}{1-\frac{\beta_T^2}{\omega^2}} & 0 & 0 \\ 0 & 1 + \frac{j\Gamma_e}{1-X_e^2} + \frac{j\Gamma_h}{1-X_h^2} & \frac{\Gamma_e X_e}{1-X_e^2} - \frac{\Gamma_h X_h}{1-X_h^2} \\ 0 & \frac{\Gamma_h X_h}{1-X_h^2} - \frac{\Gamma_e X_e}{1-X_e^2} & 1 + \frac{j\Gamma_e}{1-X_e^2} + \frac{j\Gamma_h}{1-X_h^2} \end{bmatrix}. \tag{5.209}
$$

In a more simpler form eq. (5.209) can be written as

$$
A = \begin{bmatrix} A_{xx} & 0 & 0 \\ 0 & A_{yy} & A_{yz} \\ 0 & -A_{yz} & A_{yy} \end{bmatrix}, \tag{5.210}
$$

the inverse of which is

$$A^{-1} = \begin{bmatrix} \frac{1}{A_{xx}} & 0 & 0 \\ 0 & \frac{A_{yy}}{A_{yy}^2+A_{yz}^2} & -\frac{A_{yz}}{A_{yy}^2+A_{yz}^2} \\ 0 & \frac{A_{yz}}{A_{yy}^2+A_{yz}^2} & \frac{A_{yy}}{A_{yy}^2+A_{yz}^2} \end{bmatrix}. \tag{5.211}$$

Now, $\frac{\omega_{pe}^2}{\omega^2} A^{-1} M_e^{-1}$ and $\frac{\omega_{ph}^2}{\omega^2} A^{-1} M_h^{-1}$ can be evaluated as

$$\frac{\omega_{pe}^2}{\omega^2} A^{-1} M_e^{-1} = \begin{bmatrix} \frac{1-\frac{\beta_T^2}{\omega^2}k_x^2}{1-\frac{\beta_T^2}{\omega^2}k_x^2+j(\Gamma_e+\Gamma_h)} & 0 & 0 \\ 0 & \frac{A_{yy}}{A_{yy}^2+A_{yz}^2} & -\frac{A_{yz}}{A_{yy}^2+A_{yz}^2} \\ 0 & \frac{A_{yz}}{A_{yy}^2+A_{yz}^2} & \frac{A_{yy}}{A_{yy}^2+A_{yz}^2} \end{bmatrix} \begin{bmatrix} \frac{\frac{\omega_{pe}^2}{\omega^2}}{1-\frac{\beta_T^2}{\omega^2}k_x^2} & 0 & 0 \\ 0 & \frac{\omega_{pe}^2}{1-X_e^2} & \frac{-jX_e\omega_{pe}^2}{\omega^2(1-X_e^2)} \\ 0 & \frac{jX_e\omega_{pe}^2}{\omega^2(1-X_e^2)} & \frac{\frac{\omega_{pe}^2}{\omega^2}}{1-X_e^2} \end{bmatrix}, \tag{5.212}$$

and

$$\frac{\omega_{ph}^2}{\omega^2} A^{-1} M_h^{-1} = \begin{bmatrix} \frac{1-\frac{\beta_T^2}{\omega^2}k_x^2}{1-\frac{\beta_T^2}{\omega^2}k_x^2+j(\Gamma_e+\Gamma_h)} & 0 & 0 \\ 0 & \frac{A_{yy}}{A_{yy}^2+A_{yz}^2} & -\frac{A_{yz}}{A_{yy}^2+A_{yz}^2} \\ 0 & \frac{A_{yz}}{A_{yy}^2+A_{yz}^2} & \frac{A_{yy}}{A_{yy}^2+A_{yz}^2} \end{bmatrix} \begin{bmatrix} \frac{\frac{\omega_{ph}^2}{\omega^2}}{1-\frac{\beta_T^2}{\omega^2}k_x^2} & 0 & 0 \\ 0 & \frac{\omega_{ph}^2}{1-X_h^2} & \frac{jX_h\omega_{ph}^2}{\omega^2(1-X_h^2)} \\ 0 & \frac{-jX_h\omega_{ph}^2}{\omega^2(1-X_h^2)} & \frac{\frac{\omega_{ph}^2}{\omega^2}}{1-X_h^2} \end{bmatrix}. \tag{5.213}$$

Every element of the A^{-1} matrix has been written apart from A_{xx}, which has been explicitly written out since it has wave vector dependence. Then, the permittivity tensor is

$$\tilde{\epsilon}_p(k_x;\omega) = I - \frac{\omega_{pe}^2}{\omega^2} A^{-1} M_e^{-1} - \frac{\omega_{ph}^2}{\omega^2} A^{-1} M_h^{-1}. \tag{5.214}$$

The nonlocal permittivity tensor has a similar form to the case when $\vec{B}_0 \neq 0$ and $\vec{\nabla}p = 0$, i.e.,

$$\tilde{\epsilon}_p(x;\omega) = \begin{bmatrix} \epsilon_{xx}(k_x;\omega) & 0 & 0 \\ 0 & \epsilon_{yy} & \epsilon_{yz} \\ 0 & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}. \tag{5.215}$$

The procedure to find a dispersion relation will be similar to that used in the previous sections.

The vector wave equation can be written as

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E}(\vec{x}; \omega) = \omega^2 \mu_0 \epsilon_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dy' dz' \int_0^d dx' \tilde{\epsilon}_p(|x - x'|; \omega) \cdot \vec{E}(\vec{x}; \omega). \qquad (5.216)$$

To get to an algebraic equation from the above vector equation, let the spatial Fourier transform be first defined for the infinite coordinates [47], in the form of a two-dimensional transform as

$$\vec{E}(\vec{r}; \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{E}^*(x; \omega) e^{j(k_y y + k_z z)} dk_y dk_z, \qquad (5.217)$$

and the spatially dependent permittivity kernel $G_\gamma(|x - x'|)$, with the Weyl approximation [25, 47, 62] as

$$G_\gamma(|x - x'|) = \frac{j}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{w_\gamma} e^{j(k_y(y-y') + k_z(z-z'))} e^{-jw_\gamma |x - x'|} dk_y dk_z, \qquad (5.218)$$

where $w_\gamma = \sqrt{(\gamma^2 - k_y^2 - k_z^2)}$. Also, E^* is just a notation which tells that the fields have been two-dimensionally Fourier transformed and is not the conjugate of a field quantity.

Simplifying the vector equation through the identity $\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$, we get

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}(x, y, z; \omega)) - \nabla^2 \vec{E}(x, y, z; \omega) = \omega^2 \mu_0 \epsilon_0$$
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dy' dz' \int_0^d dx' \tilde{\epsilon}_p(|x - x'|; \omega) \cdot \vec{E}(x, y, z; \omega). \qquad (5.219)$$

We can then apply the above mentioned two-dimensional Fourier transform on $\vec{E}$ and the Weyl approximation formula for the kernel G_γ of the tensor element ϵ_{xx} to get the three components of the wave equation as

$$\hat{x} \Rightarrow \frac{d}{dx}\left(\frac{d}{dx} E_x^*(x, k_y, k_z; \omega) + j k_y E_y^*(x, k_y, k_z; \omega) + j k_z E_z^*(x, k_y, k_z; \omega)\right)$$
$$-\left(\frac{d^2}{d^2 x} - k_y^2 - k_z^2\right) E_x^*(x, k_y, k_z; \omega) = \omega^2 \mu_0 \epsilon_0 \int_0^d \delta(x - x') E_x^*(x', k_y, k_z; \omega) dx'$$
$$+ \frac{j \chi k_0^2}{4\pi^2} \int_0^d \frac{e^{-jw_\gamma |x - x'|}}{w_\gamma} E_x^*(x', k_y, k_z; \omega) dx', \qquad (5.220)$$

$$\hat{y} \Rightarrow jk_y(\frac{d}{dx}E_x^*(x,k_y,k_z;\omega) + jk_yE_y^*(x,k_y,k_z;\omega) + jk_zE_z^*(x,k_y,k_z;\omega))$$

$$-(\frac{d^2}{d^2x} - k_y^2 - k_z^2)E_y^*(x,k_y,k_z;\omega) = k_0^2(\epsilon_{yy}E_y^*(x,k_y,k_z;\omega) + \epsilon_{yz}E_z^*(x,k_y,k_z;\omega)),$$

$$(5.221)$$

and

$$\hat{z} \Rightarrow jk_z(\frac{d}{dx}E_x^*(x,k_y,k_z;\omega) + jk_yE_y^*(x,k_y,k_z;\omega) + jk_z^*E_z(x,k_y,k_z;\omega))$$

$$-(\frac{d^2}{d^2x} - k_y^2 - k_z^2)E_z^*(x,k_y,k_z;\omega) = k_0^2(\epsilon_{zy}E_y^*(x,k_y,k_z;\omega) + \epsilon_{zz}E_z^*(x,k_y,k_z;\omega)),$$

$$(5.222)$$

where the partial x differentials are now total derivatives, since the other spatial and temporal dependencies have been taken care of through the use Fourier transforms. Also, the triple integral on the right hand side of y and z components has not been written since the permittivity elements are local spatially. The first integral on the right hand side of the x component is just k_0^2, but the second function has an x dependence in the kernel. Since the integral cannot be evaluated analytically, we need to operate upon the kernel in such a way that gives us a delta function. This can be done if we multiply $\frac{e^{-jw_\gamma|x-x'|}}{w_\gamma}$ by $\frac{d^2}{d^2x} + w_\gamma^2$ [24, 47]. This gives

$$(\frac{d^2}{d^2x} + w_\gamma^2)(\frac{e^{-jw_\gamma|x-x'|}}{w_\gamma}) = 2\pi j\delta(x - x').$$

$$(5.223)$$

Using this operation on eq. (5.220), we can then get rid of the integral to get

$$(\frac{d^2}{d^2x} + w_\gamma^2)(\frac{d}{dx}(\frac{d}{dx}E_x^*(x,k_y,k_z;\omega) + jk_yE_y^*(x,k_y,k_z;\omega) + jk_zE_z^*(x,k_y,k_z;\omega))$$

$$-(\frac{d^2}{d^2x} - k_y^2 - k_z^2 + k_0^2)E_x^*(x,k_y,k_z;\omega)) = -\frac{\chi k_0^2}{2\pi}E_x^*(x,k_y,k_z;\omega).$$

$$(5.224)$$

The above derivation, in a sense, means that the following field form has been assumed $\vec{E}(x,y,z;\omega) = \vec{E}^*(x)e^{j(k_yy+k_zz-\omega t)}$ and to get to an algebraic equation, we need a solution in x that is a valid surface wave mode. We already have a solution which is of the form

$\vec{E}^*(x) = \vec{E}_0 e^{-\alpha x}$ [32]. Substituting this form back into the wave equations, we would get the desired algebraic form for a surface wave. So, the wave equations can be written in algebraic form as

$$\hat{x} \Rightarrow (\alpha^2 + w_\gamma^2)(-\alpha(-\alpha E_{0x} + jk_y E_{0y} + jk_z E_{0z}) - (\alpha^2 - k_y^2 - k_z^2 + k_0^2)E_{0x}$$
$$= -\frac{\chi k_0^2}{2\pi}E_{0x}, (5.225)$$

$$\hat{y} \Rightarrow jk_y(-\alpha E_{0x} + jk_y E_{0y} + jk_z E_{0z}) - (\alpha^2 - k_y^2 - k_z^2)E_{0y} = k_0^2(\epsilon_{yy}E_{0y} + \epsilon_{yz}E_{0z}), (5.226)$$

$$\hat{z} \Rightarrow jk_z(-\alpha E_{0x} + jk_y E_{0y} + jk_z E_{0z}) - (\alpha^2 - k_y^2 - k_z^2)E_{0z} = k_0^2\epsilon_{zy}(E_{0y} + \epsilon_{zz}E_{0z}), (5.227)$$

and only the constants $\vec{E}_0$ are needed to be found. Having derived these equations, now we can assume propagation in y or z directions to simplify the equations further. Therefore, Voigt($\vec{k} \perp \vec{B}_0$) and Faraday ($\vec{k} \parallel \vec{B}_0$) geometries will be analyzed in the next two sections.

5.3.1 Voigt Geometry with $\vec{B}_0$ Perpendicular to the Interface and $\vec{\nabla}p \neq 0$

When $\vec{k} \perp \vec{B}_0$, for Voigt geometry, we have the liberty of choosing the propagation direction, since both k_y and k_z are perpendicular to the steady magnetic field [12]. Staying with the continuity of the analysis done previously, the propagation direction is assumed in y direction. Therefore, in the wave equation derived in eqs. (5.225), (5.226), and (5.227)

$$\hat{x} \Rightarrow (\alpha^2 + w_\gamma^2)(-\alpha(-\alpha E_{0x} + jk_y E_{0y} + jk_z E_{0z}) - (\alpha^2 - k_y^2 - k_z^2 + k_0^2)E_{0x}$$
$$= -\frac{\chi k_0^2}{2\pi}E_{0x}, (5.228)$$

$$\hat{y} \Rightarrow jk_y(-\alpha E_{0x} + jk_y E_{0y} + jk_z E_{0z}) - (\alpha^2 - k_y^2 - k_z^2)E_{0y} = k_0^2(\epsilon_{yy}E_{0y} + \epsilon_{yz}E_{0z}), (5.229)$$

$$\hat{z} \Rightarrow jk_z(-\alpha E_{0x} + jk_y E_{0y} + jk_z E_{0z}) - (\alpha^2 - k_y^2 - k_z^2)E_{0z} = k_0^2\epsilon_{zy}(E_{0y} + \epsilon_{zz}E_{0z}), (5.230)$$

substituting $k_z = 0$, we get

$$\hat{x} \Rightarrow (\alpha^2 + w_\gamma^2)(-\alpha(-\alpha E_{0x} + jk_y E_{0y}) - (\alpha^2 - k_y^2 + k_0^2)E_{0x}$$
$$= -\frac{\chi k_0^2}{2\pi}E_{0x}, \qquad (5.231)$$

$$\hat{y} \Rightarrow jk_y(-\alpha E_{0x} + jk_y E_{0y}) - (\alpha^2 - k_y^2)E_{0y} = k_0^2(\epsilon_{yy}E_{0y} + \epsilon_{yz}E_{0z}), \tag{5.232}$$

and

$$\hat{z} \Rightarrow -(\alpha^2 - k_y^2)E_{0z} = k_0^2(\epsilon_{zy}E_{0y} + \epsilon_{zz}E_{0z}). \tag{5.233}$$

Homogenizing the equations by combining terms with the same coefficients $\vec{E}_0$, we get three algebraic equations

$$[(\alpha^2 + w_\gamma^2)(k_y^2 - k_0^2) + \frac{\chi k_0^2}{2\pi}]E_{0x} - jk_y\alpha(\alpha^2 + w_\gamma^2)E_{0y} = 0, \tag{5.234}$$

$$jk_y\alpha E_{0x} + (\alpha^2 + k_0^2\epsilon_{yy})E_{0y} + k_0^2\epsilon_{yz}E_{0z} = 0, \tag{5.235}$$

$$k_0^2\epsilon_{zy}E_{0y} + (\alpha^2 - k_y^2 + k_0^2\epsilon_{zz})E_{0z} = 0. \tag{5.236}$$

We can see from eqs. (5.234), (5.235), and (5.236) that all the field quantities are related because of cross terms in the permittivity tensor. Having found these equations, they can be reduced to two by substituting one of the coefficients in terms of the other two [12]. From eq. (5.236), we have

$$E_{0z} = -\frac{k_0^2\epsilon_{zy}}{\alpha^2 - k_y^2 + k_0^2\epsilon_{zz}}E_{0y}. \tag{5.237}$$

Substituting eq. (5.237) into eq. (5.235), we get

$$jk_y\alpha E_{0x} + (\alpha^2 + k_0^2\epsilon_{yy})E_{0y} - \frac{k_0^4\epsilon_{zy}\epsilon_{yz}}{\alpha^2 - k_y^2 + k_0^2\epsilon_{zz}}E_{0y} = 0, \tag{5.238}$$

$$\Rightarrow jk_y\alpha(\alpha^2 - k_y^2 + k_0^2\epsilon_{zz})E_{0x} + [(\alpha^2 + k_0^2\epsilon_{yy})(\alpha^2 - k_y^2 + k_0^2\epsilon_{zz})$$
$$-k_0^4\epsilon_{zy}\epsilon_{yz}]E_{0y} = 0, \tag{5.239}$$

which, along with the x component of the wave equation,

$$[(\alpha^2 + w_\gamma^2)(k_y^2 - k_0^2) + \frac{\chi k_0^2}{2\pi}]E_{0x} - jk_y\alpha(\alpha^2 + w_\gamma^2)E_{0y} = 0, \tag{5.240}$$

gives us two linear algebraic equations, which can be solved for through the procedure adopted before.

Writing the algebraic equations as

$$A(\alpha)E_{0x} + B(\alpha)E_{0y} = 0, \tag{5.241}$$

$$C(\alpha)E_{0x} + D(\alpha)E_{0y} = 0, \tag{5.242}$$

where

$$A(\alpha) = (\alpha^2 + w_\gamma^2)(k_y^2 - k_0^2) + \frac{\chi k_0^2}{2\pi}, \tag{5.243}$$

$$B(\alpha) = -jk_y\alpha(\alpha^2 + w_\gamma^2), \tag{5.244}$$

$$C(\alpha) = jk_y\alpha(\alpha^2 - k_y^2 + k_0^2\epsilon_{zz}), \tag{5.245}$$

$$D(\alpha) = (\alpha^2 + k_0^2\epsilon_{yy})(\alpha^2 - k_y^2 + k_0^2\epsilon_{zz}) - k_0^4\epsilon_{zy}\epsilon_{yz}, \tag{5.246}$$

equating the secular determinant of the matrix equation gives the number of roots on the x dependence, i.e.,

$$\begin{vmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{vmatrix} = 0, \tag{5.247}$$

$$\Rightarrow A(\alpha)D(\alpha) - B(\alpha)C(\alpha) = 0. \tag{5.248}$$

Although, the actual values of the roots are not known till now, one can predict their number by looking at the power of α in AD and BC. Since A has a power of 2, B of 3, C of 3, and D of 4, $AD - BC$ will be a polynomial equation of power 6 in α. The solution for E_{0x} and E_{0y} can then be either

$$E_{0x} = \sum_i F_i B(\alpha_i), \tag{5.249}$$

$$E_{0y} = -\sum_i F_i A(\alpha_i), \tag{5.250}$$

or

$$E_{0x} = \sum_i F_i D(\alpha_i), \tag{5.251}$$

$$E_{0y} = -\sum_i F_i C(\alpha_i). \tag{5.252}$$

Going with the first solution, i.e., eqs. (5.249) and (5.250), the full solutions for E_{x1} and E_{y1} are given as

$$E_{x1}(x, y; \omega) = \sum_i F_i B(\alpha_i) e^{-\alpha_i x} e^{j(k_y y - \omega t)}, \tag{5.253}$$

$$E_{y1}(x, y; \omega) = -\sum_i F_i A(\alpha_i) e^{-\alpha_i x} e^{j(k_y y - \omega t)}, \tag{5.254}$$

and using eq. (5.237), we get

$$E_{z1}(x, y; \omega) = \sum_i \frac{k_0^2 \epsilon_{zy}}{\alpha_i^2 - k_y^2 + k_0^2 \epsilon_{zz}} F_i A(\alpha_i) e^{-\alpha_i x} e^{j(k_y y - \omega t)}. \tag{5.255}$$

In medium 2, i.e., free space, we need E_{y2} and E_{z2}, the functional forms of which are

$$E_{y2}(x, y; \omega) = E_{1y} e^{-\beta x} e^{j(k_y y - \omega t)}, \tag{5.256}$$

$$E_{z2}(x, y; \omega) = E_{1z} e^{-\beta x} e^{j(k_y y - \omega t)}, \tag{5.257}$$

where β is the positive root of the wave equation $\nabla^2 \vec{E}_2 + k_0^2 \epsilon_r \vec{E}_2 = 0$, i.e., $\beta = \sqrt{(k_y^2 - k_0^2 \epsilon_r)}$. The third component, E_{x2} is determined through the use of the divergence equation $\vec{\nabla} \cdot \vec{E}_2 = 0$, which gives

$$E_{1x} = \frac{jk_y}{\beta} E_{1y}. \tag{5.258}$$

So, knowing the three electric field components in the substrate and in free space, the remaining magnetic field components in the substrate and in free space can be readily found. To find a unique dispersion relation, we have to have eight conditions for eight unknowns ($F_1, F_2, F_3, F_4, F_5, F_6, E_{1y}$, and E_{1z}). We have eight boundary conditions in total (two on tangential electric fields at $x = 0$, two on continuity of tangential electric fields

at $x = d$, two on continuity of tangential magnetic fields at $x = d$, and the continuity of normal $\vec{D}$ and $\vec{B}$) [52]. Finding the determinant of an 8×8 matrix will not be easy and finding the roots of the resulting transcendental equation will also be hectic. So, as done before, noting that $\alpha_1 = -\alpha_2$, $\alpha_3 = -\alpha_4$, and $\alpha_5 = -\alpha_6$, the number of unknowns can be reduced to five. We can then use the electric field tangential boundary conditions at $x = 0$, continuity of tangential electric fields at $x = d$, continuity of tangential magnetic fields at $x = d$, and the continuity of normal $\vec{B}$, at $x = d$ can be used. Since there are more boundary conditions than the number of unknowns, we have some redundancy [12,31]. Therefore, only tangential field boundary conditions will be utilized.

Going back to the field solutions in the substrate, i.e.,

$$E_{x1}(x, y; \omega) = \sum_i F_i B(\alpha_i) e^{-\alpha_i x} e^{j(k_y y - \omega t)}, \tag{5.259}$$

$$E_{y1}(x, y; \omega) = -\sum_i F_i A(\alpha_i) e^{-\alpha_i x} e^{j(k_y y - \omega t)}, \tag{5.260}$$

$$E_{z1}(x, y; \omega) = \sum_i \frac{k_0^2 \epsilon_{zy}}{\alpha_i^2 - k_y^2 + k_0^2 \epsilon_{zz}} F_i A(\alpha_i) e^{-\alpha_i x} e^{j(k_y y - \omega t)}. \tag{5.261}$$

With the mentioned simplifications, eqs. (5.259), (5.260), and (5.261) are reduced to

$$
\begin{aligned}
E_{x1}(x, y; \omega) = {} & [F_1(B(\alpha_1)e^{-\alpha_1 x} + B(-\alpha_1)e^{\alpha_1 x}) \\
& + F_3(B(\alpha_3)e^{-\alpha_3 x} + B(-\alpha_3)e^{\alpha_3 x}) \\
& + F_5(B(\alpha_5)e^{-\alpha_5 x} + B(-\alpha_5)e^{\alpha_5 x})]e^{j(k_y y - \omega t)}, \tag{5.262}
\end{aligned}
$$

$$
\begin{aligned}
E_{y1}(x, y; \omega) = {} & -[F_1(A(\alpha_1)e^{-\alpha_1 x} + A(-\alpha_1)e^{\alpha_1 x}) \\
& + F_3(A(\alpha_3)e^{-\alpha_3 x} + A(-\alpha_3)e^{\alpha_3 x}) \\
& + F_5(A(\alpha_5)e^{-\alpha_5 x} + A(-\alpha_5)e^{\alpha_5 x})]e^{j(k_y y - \omega t)}, \tag{5.263}
\end{aligned}
$$

and

$$
\begin{aligned}
E_{z1}(x,y;\omega) &= [F_1(\frac{k_0^2\epsilon_{zy}}{\alpha_1^2 - k_y^2 + k_0^2\epsilon_{zz}}A(\alpha_1)e^{-\alpha_1 x} + \frac{k_0^2\epsilon_{zy}}{\alpha_1^2 - k_y^2 + k_0^2\epsilon_{zz}}A(-\alpha_1)e^{\alpha_1 x}) \\
&+ F_3(\frac{k_0^2\epsilon_{zy}}{\alpha_3^2 - k_y^2 + k_0^2\epsilon_{zz}}A(\alpha_3)e^{-\alpha_3 x} + \frac{k_0^2\epsilon_{zy}}{\alpha_3^2 - k_y^2 + k_0^2\epsilon_{zz}}A(-\alpha_3)e^{\alpha_3 x}) \\
&+ F_5(\frac{k_0^2\epsilon_{zy}}{\alpha_5^2 - k_y^2 + k_0^2\epsilon_{zz}}A(\alpha_5)e^{-\alpha_5 x} \\
&+ \frac{k_0^2\epsilon_{zy}}{\alpha_5^2 - k_y^2 + k_0^2\epsilon_{zz}}A(-\alpha_5)e^{\alpha_5 x})]e^{j(k_y y - \omega t)}.
\end{aligned} \tag{5.264}
$$

From the above equations, we can derive the three magnetic field components in medium 1, with the help of the curl equation $\vec{\nabla} \times \vec{E}_1 = j\omega\mu_0\vec{H}_1$, the components of which are

$$
\hat{x} \Rightarrow jk_y E_{0z} = j\omega\mu_0 H_{0x}, \tag{5.265}
$$

$$
\hat{y} \Rightarrow \alpha E_{0z} = j\omega\mu_0 H_{0y}, \tag{5.266}
$$

$$
\hat{z} \Rightarrow -\alpha E_{0y} - jk_y E_{0x} = j\omega\mu_0 H_{0z}, \tag{5.267}
$$

or in simple form, as

$$
H_{0x} = \frac{k_y}{\omega\mu_0}E_{0z}, \tag{5.268}
$$

$$
H_{0y} = \frac{\alpha}{j\omega\mu_0}E_{0z}, \tag{5.269}
$$

$$
H_{0z} = -\frac{1}{j\omega\mu_0}(\alpha E_{0y} + jk_y E_{0x}). \tag{5.270}
$$

The full solutions can be written using eqs. (5.268), (5.269), and (5.270) as

$$
\begin{aligned}
H_{x1}(x,y;\omega) &= \frac{k_y}{\omega\mu_0}[F_1(\frac{k_0^2\epsilon_{zy}}{\alpha_1^2 - k_y^2 + k_0^2\epsilon_{zz}}A(\alpha_1)e^{-\alpha_1 x} + \frac{k_0^2\epsilon_{zy}}{\alpha_1^2 - k_y^2 + k_0^2\epsilon_{zz}}A(-\alpha_1)e^{\alpha_1 x}) \\
&+ F_3(\frac{k_0^2\epsilon_{zy}}{\alpha_3^2 - k_y^2 + k_0^2\epsilon_{zz}}A(\alpha_3)e^{-\alpha_3 x} + \frac{k_0^2\epsilon_{zy}}{\alpha_3^2 - k_y^2 + k_0^2\epsilon_{zz}}A(-\alpha_3)e^{\alpha_3 x}) \\
&+ F_5(\frac{k_0^2\epsilon_{zy}}{\alpha_5^2 - k_y^2 + k_0^2\epsilon_{zz}}A(\alpha_5)e^{-\alpha_5 x} \\
&+ \frac{k_0^2\epsilon_{zy}}{\alpha_5^2 - k_y^2 + k_0^2\epsilon_{zz}}A(-\alpha_5)e^{\alpha_5 x})]e^{j(k_y y - \omega t)},
\end{aligned} \tag{5.271}
$$

$$H_{y1}(x,y;\omega) = [F_1(\frac{\alpha_1 k_0^2 \epsilon_{zy}}{j\omega\mu_0(\alpha_1^2 - k_y^2 + k_0^2\epsilon_{zz})}A(\alpha_1)e^{-\alpha_1 x}$$

$$+\frac{-\alpha_1 k_0^2 \epsilon_{zy}}{j\omega\mu_0(\alpha_1^2 - k_y^2 + k_0^2\epsilon_{zz})}A(-\alpha_1)e^{\alpha_1 x})$$

$$+F_3(\frac{\alpha_3 k_0^2 \epsilon_{zy}}{j\omega\mu_0(\alpha_3^2 - k_y^2 + k_0^2\epsilon_{zz})}A(\alpha_3)e^{-\alpha_3 x}$$

$$+\frac{-\alpha_3 k_0^2 \epsilon_{zy}}{j\omega\mu_0(\alpha_3^2 - k_y^2 + k_0^2\epsilon_{zz})}A(-\alpha_3)e^{\alpha_3 x})$$

$$+F_5(\frac{\alpha_5 k_0^2 \epsilon_{zy}}{j\omega\mu_0(\alpha_5^2 - k_y^2 + k_0^2\epsilon_{zz})}A(\alpha_5)e^{-\alpha_5 x}$$

$$+\frac{-\alpha_5 k_0^2 \epsilon_{zy}}{j\omega\mu_0(\alpha_5^2 - k_y^2 + k_0^2\epsilon_{zz})}A(-\alpha_5)e^{\alpha_5 x})]e^{j(k_y y - \omega t)}, \qquad (5.272)$$

and

$$H_{z1}(x,y;\omega) = -\frac{1}{j\omega\mu_0}[(F_1(\alpha_1 A(\alpha_1)e^{-\alpha_1 x} - \alpha_1 A(-\alpha_1)e^{\alpha_1 x})$$

$$+F_3(\alpha_3 A(\alpha_3)e^{-\alpha_3 x} - \alpha_3 A(-\alpha_3)e^{\alpha_3 x})$$

$$+F_5(\alpha_5 A(\alpha_5)e^{-\alpha_5 x} - \alpha_5 A(-\alpha_5)e^{\alpha_5 x})$$

$$+jk_y(F_1(B(\alpha_1)e^{-\alpha_1 x} + B(-\alpha_1)e^{\alpha_1 x})$$

$$+F_3(B(\alpha_3)e^{-\alpha_3 x} + B(-\alpha_3)e^{\alpha_3 x}) + F_5(B(\alpha_5)e^{-\alpha_5 x}$$

$$+B(-\alpha_5)e^{\alpha_5 x})))]e^{j(k_y y - \omega t)}. \qquad (5.273)$$

In the above equations, care needs to be taken while substituting the coefficients which have α dependence.

Using the same Maxwell's curl equation $\vec{\nabla} \times \vec{E}_2 = j\omega\mu_0\vec{H}_2$ in medium 2, the magnetic field components on the free space side can be found similarly [2,8]. They are

$$H_{1x} = \frac{k_y}{\omega\mu_0}E_{1z}, \qquad (5.274)$$

$$H_{1y} = \frac{\beta}{j\omega\mu_0}E_{1z}, \qquad (5.275)$$

$$H_{1z} = \frac{-\beta E_{1y} - jk_y E_{1x}}{j\omega\mu_0}, \qquad (5.276)$$

$$\Rightarrow H_{1z} = \frac{-\beta^2 + k_y^2}{j\beta\omega\mu_0}E_{1y}, \qquad (5.277)$$

since $E_{1x} = \frac{jk_y}{\beta}E_{1y}$ in the third equation. Having found the coefficients in terms of E_{1y} and E_{1z}, the full solution for the magnetic field components is given as

$$H_{x2}(x, y; \omega) = \frac{k_y}{\omega\mu_0}E_{1z}e^{-\beta x}e^{j(k_yy-\omega t)}, \tag{5.278}$$

$$H_{y2}(x, y; \omega) = \frac{\beta}{j\omega\mu_0}E_{1z}e^{-\beta x}e^{j(k_yy-\omega t)}, \tag{5.279}$$

$$H_{z2}(x, y; \omega) = \frac{-\beta^2 + k_y^2}{j\beta\omega\mu_0}E_{1y}e^{-\beta x}e^{j(k_yy-\omega t)}. \tag{5.280}$$

Having found all the fields in medium 1 and 2, we can apply the aforesaid boundary condition on E_{y1} at the ground plane $(x = 0)$ and at the interface $(x = d)$ to get

$$-[F_1(A(\alpha_1) + A(-\alpha_1)) + F_3(A(\alpha_3) + A(-\alpha_3)) + F_5(A(\alpha_5) + A(-\alpha_5))] = 0, \tag{5.281}$$

$$-[F_1(A(\alpha_1)e^{-\alpha_1 d} + A(-\alpha_1)e^{\alpha_1 d}) + F_3(A(\alpha_3)e^{-\alpha_3 d} + A(-\alpha_3)e^{\alpha_3 d})$$

$$+F_5(A(\alpha_5)e^{-\alpha_5 d} + A(-\alpha_5)e^{\alpha_5 d})] = E_{1y}e^{-\beta d}, \tag{5.282}$$

$$F_1(\frac{k_0^2\epsilon_{zy}}{\alpha_1^2 - k_y^2 + k_0^2\epsilon_{zz}}A(\alpha_1)e^{-\alpha_1 d} + \frac{k_0^2\epsilon_{zy}}{\alpha_1^2 - k_y^2 + k_0^2\epsilon_{zz}}A(-\alpha_1)e^{\alpha_1 d})$$

$$+F_3(\frac{k_0^2\epsilon_{zy}}{\alpha_3^2 - k_y^2 + k_0^2\epsilon_{zz}}A(\alpha_3)e^{-\alpha_3 d} + \frac{k_0^2\epsilon_{zy}}{\alpha_3^2 - k_y^2 + k_0^2\epsilon_{zz}}A(-\alpha_3)e^{\alpha_3 d})$$

$$+F_5(\frac{k_0^2\epsilon_{zy}}{\alpha_5^2 - k_y^2 + k_0^2\epsilon_{zz}}A(\alpha_5)e^{-\alpha_5 d} + \frac{k_0^2\epsilon_{zy}}{\alpha_5^2 - k_y^2 + k_0^2\epsilon_{zz}}A(-\alpha_5)e^{\alpha_5 d}) = E_{1z}e^{-\beta d}, \tag{5.283}$$

$$F_1(\frac{\alpha_1 k_0^2\epsilon_{zy}}{j\omega\mu_0(\alpha_1^2 - k_y^2 + k_0^2\epsilon_{zz})}A(\alpha_1)e^{-\alpha_1 d} + \frac{-\alpha_1 k_0^2\epsilon_{zy}}{j\omega\mu_0(\alpha_1^2 - k_y^2 + k_0^2\epsilon_{zz})}A(-\alpha_1)e^{\alpha_1 d})$$

$$+F_3(\frac{\alpha_3 k_0^2\epsilon_{zy}}{j\omega\mu_0(\alpha_3^2 - k_y^2 + k_0^2\epsilon_{zz})}A(\alpha_3)e^{-\alpha_3 d} + \frac{-\alpha_3 k_0^2\epsilon_{zy}}{j\omega\mu_0(\alpha_3^2 - k_y^2 + k_0^2\epsilon_{zz})}A(-\alpha_3)e^{\alpha_3 d})$$

$$+F_5(\frac{\alpha_5 k_0^2\epsilon_{zy}}{j\omega\mu_0(\alpha_5^2 - k_y^2 + k_0^2\epsilon_{zz})}A(\alpha_5)e^{-\alpha_5 d}$$

$$+\frac{-\alpha_5 k_0^2\epsilon_{zy}}{j\omega\mu_0(\alpha_5^2 - k_y^2 + k_0^2\epsilon_{zz})}A(-\alpha_5)e^{\alpha_5 d}) = \frac{\beta}{j\omega\mu_0}E_{1z}e^{-\beta d}, \tag{5.284}$$

$$-\frac{1}{j\omega\mu_0}[(F_1(\alpha_1 A(\alpha_1)e^{-\alpha_1 d} - \alpha_1 A(-\alpha_1)e^{\alpha_1 d}) + F_3(\alpha_3 A(\alpha_3)e^{-\alpha_3 d} - \alpha_3 A(-\alpha_3)e^{\alpha_3 d})$$

$$+F_5(\alpha_5 A(\alpha_5)e^{-\alpha_5 d} - \alpha_5 A(-\alpha_5)e^{\alpha_5 d}) + jk_y(F_1(B(\alpha_1)e^{-\alpha_1 d} + B(-\alpha_1)e^{\alpha_1 d})$$

$$+F_3(B(\alpha_3)e^{-\alpha_3 d} + B(-\alpha_3)e^{\alpha_3 d})$$

$$+F_5(B(\alpha_5)e^{-\alpha_5 d} + B(-\alpha_5)e^{\alpha_5 d})))] = \frac{-\beta^2 + k_y^2}{j\beta\omega\mu_0}E_{1y}e^{-\beta d}. \tag{5.285}$$

In the above derivations, it has been assumed that both the semiconductor and free space are nonmagnetic (they have the same permeability, μ_0, which gets cancelled). The above equations can then be written in matrix form, so that the determinant of the matrix gives a dispersion relation

$$
\begin{bmatrix}
A_1 & A_3 & A_5 & 0 & 0 \\
J_1 & J_3 & J_5 & e^{-\beta d} & 0 \\
K_1 & K_3 & K_5 & 0 & -e^{-\beta d} \\
L_1 & L_3 & L_5 & 0 & -\frac{\beta}{j\omega\mu_0}e^{-\beta d} \\
M_1 & M_3 & M_5 & -\frac{k_y^2-\beta^2}{j\beta\omega\mu_0}e^{-\beta d} & 0
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_3 \\
F_5 \\
E_{1y} \\
E_{1z}
\end{bmatrix}
= 0,
\tag{5.286}
$$

where

$$
A_1 = A(\alpha_1) + A(-\alpha_1),
\tag{5.287}
$$

$$
A_3 = A(\alpha_3) + A(-\alpha_3),
\tag{5.288}
$$

$$
A_5 = A(\alpha_5) + A(-\alpha_5),
\tag{5.289}
$$

$$
J_1 = A(\alpha_1)e^{-\alpha_1 d} + A(-\alpha_1)e^{\alpha_1 d},
\tag{5.290}
$$

$$
J_3 = A(\alpha_3)e^{-\alpha_3 d} + A(-\alpha_3)e^{\alpha_3 d},
\tag{5.291}
$$

$$
J_5 = A(\alpha_5)e^{-\alpha_5 d} + A(-\alpha_5)e^{\alpha_5 d},
\tag{5.292}
$$

$$
K_1 = \frac{k_0^2 \epsilon_{zy}}{\alpha_1^2 - k_y^2 + k_0^2 \epsilon_{zz}}[A(\alpha_1)e^{-\alpha_1 d} + A(-\alpha_1)e^{\alpha_1 d}],
\tag{5.293}
$$

$$
K_3 = \frac{k_0^2 \epsilon_{zy}}{\alpha_3^2 - k_y^2 + k_0^2 \epsilon_{zz}}[A(\alpha_3)e^{-\alpha_3 d} + A(-\alpha_3)e^{\alpha_3 d}],
\tag{5.294}
$$

$$
K_5 = \frac{k_0^2 \epsilon_{zy}}{\alpha_5^2 - k_y^2 + k_0^2 \epsilon_{zz}}[A(\alpha_5)e^{-\alpha_5 d} + A(-\alpha_5)e^{\alpha_5 d}],
\tag{5.295}
$$

$$L_1 = \frac{\alpha_1 k_0^2 \epsilon_{zy}}{j\omega\mu_0(\alpha_1^2 - k_y^2 + k_0^2 \epsilon_{zz})}[A(\alpha_1)e^{-\alpha_1 d} - A(-\alpha_1)e^{\alpha_1 d}], \tag{5.296}$$

$$L_3 = \frac{\alpha_3 k_0^2 \epsilon_{zy}}{j\omega\mu_0(\alpha_3^2 - k_y^2 + k_0^2 \epsilon_{zz})}[A(\alpha_3)e^{-\alpha_3 d} - A(-\alpha_3)e^{\alpha_3 d}], \tag{5.297}$$

$$L_5 = \frac{\alpha_5 k_0^2 \epsilon_{zy}}{j\omega\mu_0(\alpha_5^2 - k_y^2 + k_0^2 \epsilon_{zz})}[A(\alpha_5)e^{-\alpha_5 d} - A(-\alpha_5)e^{\alpha_5 d}], \tag{5.298}$$

and

$$M_1 = -\frac{1}{j\omega\mu_0}[\alpha_1 A(\alpha_1)e^{-\alpha_1 d} - \alpha_1 A(-\alpha_1)e^{\alpha_1 d} + jk_y(B(\alpha_1)e^{-\alpha_1 d} \tag{5.299}$$
$$+ B(-\alpha_1)e^{\alpha_1 d})],$$

$$M_3 = -\frac{1}{j\omega\mu_0}[\alpha_3 A(\alpha_3)e^{-\alpha_3 d} - \alpha_3 A(-\alpha_3)e^{\alpha_3 d} + jk_y(B(\alpha_3)e^{-\alpha_3 d} \tag{5.300}$$
$$+ B(-\alpha_3)e^{\alpha_3 d})],$$

$$M_5 = -\frac{1}{j\omega\mu_0}[\alpha_5 A(\alpha_5)e^{-\alpha_5 d} - \alpha_5 A(-\alpha_5)e^{\alpha_5 d} + jk_y(B(\alpha_5)e^{-\alpha_5 d} \tag{5.301}$$
$$+ B(-\alpha_5)e^{\alpha_5 d})].$$

The secular determinant of eq. (5.286) will give us a dispersion relation. Also, since all the field components are present and are interdependent, this mode is a hybrid mode as shown in fig. 5.5.

5.3.2   Results for Voigt Geometry

When the dispersion relation eq. (5.286) was input into the GA toolbox, it did not give any result. There might an issue with the resolution or the complexity of the equation. Further tests will have be done to ensure that GA runs well.

5.3.3   Faraday Geometry ($\vec{k} \parallel \vec{B}_0$) with $\vec{B}_0$ Perpendicular to the Interface

For the surface wave analysis, this geometry is of no use, since $\vec{k} \parallel \vec{B}_0$, but $\vec{B}_0 = \hat{x}B_0$, implying that no traveling waves exist in the x direction [12, 49]. So, as done previously, this geometry needs no further analysis.
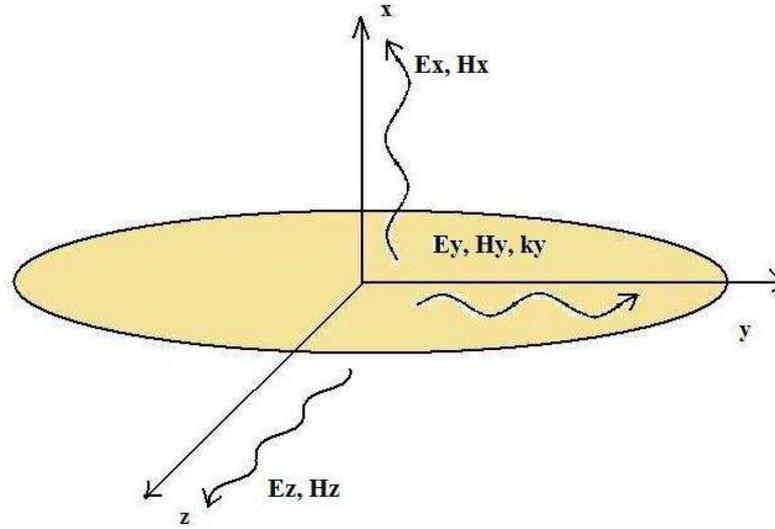
Fig. 5.5: Depiction of a hybrid mode.

5.4 $\vec{B}_0$ Parallel to the Interface

Before starting the derivation of the wave equations, the form of the permittivity tensor $\tilde{\epsilon}_p$ needs to be found. We have for an arbitrary orientation of $\vec{B}_0$, the matrices M_h and M_e as

$$M_h = \begin{bmatrix} 1 - \frac{\beta_T^2}{\omega^2}k_x^2 & 0 & \frac{j\Omega_{ch}}{\omega}sin\theta \\ 0 & 1 & -\frac{j\Omega_{ch}}{\omega}cos\theta \\ -\frac{j\Omega_{ch}}{\omega}sin\theta & \frac{j\Omega_{ch}}{\omega}cos\theta & 1 \end{bmatrix}, \tag{5.302}$$

$$M_e = \begin{bmatrix} 1 - \frac{\beta_T^2}{\omega^2}k_x^2 & 0 & -\frac{j\Omega_{ce}}{\omega}sin\theta \\ 0 & 1 & \frac{j\Omega_{ce}}{\omega}cos\theta \\ \frac{j\Omega_{ce}}{\omega}sin\theta & -\frac{j\Omega_{ce}}{\omega}cos\theta & 1 \end{bmatrix}. \tag{5.303}$$

For the case under consideration, $\theta = 90°$. With this in mind, M_h and M_e matrices can be reduced to

$$M_h = \begin{bmatrix} 1 - \frac{\beta_T^2}{\omega^2} k_x^2 & 0 & jX_h \\ 0 & 1 & 0 \\ -jX_h & 0 & 1 \end{bmatrix}, \tag{5.304}$$

$$M_e = \begin{bmatrix} 1 - \frac{\beta_T^2}{\omega^2} k_x^2 & 0 & -jX_e \\ 0 & 1 & 0 \\ jX_e & 0 & 1 \end{bmatrix}, \tag{5.305}$$

where $X_h = \frac{\Omega_{ch}}{\omega}$ and $X_e = \frac{\Omega_{ce}}{\omega}$. We want to find the tensor $\tilde{\epsilon}_p$, which is given by the equation

$$\tilde{\epsilon}_p(k_x; \omega) = I - \frac{\omega_{ph}^2}{\omega^2} M_h^{-1} - \frac{\omega_{pe}^2}{\omega^2} M_e^{-1}. \tag{5.306}$$

We need to have the inverse of M_e and M_h. They are

$$M_e^{-1} = \frac{1}{\Delta_e} \begin{bmatrix} 1 & 0 & jX_e \\ 0 & 1 - \frac{\beta_T^2}{\omega^2} k_x^2 - X_e^2 & 0 \\ -jX_e & 0 & 1 - \frac{\beta_T^2}{\omega^2} k_x^2 \end{bmatrix}, \tag{5.307}$$

and

$$M_h^{-1} = \frac{1}{\Delta_h} \begin{bmatrix} 1 & 0 & -jX_h \\ 0 & 1 - \frac{\beta_T^2}{\omega^2} k_x^2 - X_h^2 & 0 \\ jX_h & 0 & 1 - \frac{\beta_T^2}{\omega^2} k_x^2 \end{bmatrix}. \tag{5.308}$$

$\Delta_s = 1 - \frac{\beta_T^2}{\omega^2} k_x^2 - X_s^2$ is the determinant of the matrix of species s in eqs. (5.307) and (5.308). When Δ_s is multiplied into the whole matrix, element ϵ_{xx} for example, has an extra term X_s^2, which is the square of the ratio of the cyclotron frequency to the applied frequency, both the species, i.e., the holes and the electrons will have a different root γ, when they are inverted, implying that the permittivity tensor would have a different form from Agarwal

et al. [47]. So, manipulating the wave equations with the operator $\frac{d^2}{dx^2} + w_\gamma^2$, to get rid of the finite integral in x would be a difficult task, since both the species would have their own γ. This would require that both the electron and hole masses be equal at RF frequencies of interest. The masses can be assumed to be equal approximately either only at Fermi velocities (v_{fs}) or at RF frequencies. Both approximations cannot be taken into account together. So, this case will not be analyzed further, since we want to keep the motion of holes and electrons at RF frequencies to be as different as possible, implying that $m_e \neq m_h$, at RF frequencies.

Chapter 6

Conclusions and Future Work

6.1 Conclusions

Throughout the course of the thesis, emphasis has been put on the derivation of a dispersion relation that is both mathematically and physically valid, apart from being unique. Many configurations were studied starting from the most general to the more specific. Chapter 3 dealt with deriving dispersion relation for isotropic behavior of a semiconductor, in which it was found that the form of the permittivity forces the use of TE/TM decomposition. In the fourth chapter, a steady magnetic field was added to the fluid momentum equation, which rendered the permittivity of the substrate to be a tensor. Different configurations and geometries such as Voigt and Faraday were considered with different orientations of $\vec{B}_0$ and the respective dispersion relations were found. Chapter 5 considered the semiconductor medium as being effected both by anisotropy due to $\vec{B}_0$ and spatial dispersion due to $\vec{\nabla}p \neq 0$. The main procedure that was used throughout utilized the perturbation theory, which is a useful tool when dealing analytically with nonlinear equations.

6.2 Future Work

This thesis, although very abstract and theoretical, has many future ramifications. Most of this work is a buildup to more complex models of plasmas, as the fluid equations used to derive the permittivity for a semiconductor are scalable to space plasmas. As mentioned, surface waves can exist where there is an interface between two different mediums. These can exist in micro and sub-micro level active devices and can also exist in space such as in ionosphere and magnetosphere. Using magnetohydrodynamics (MHD), similar models of permittivity can be found for space plasmas and surface waves can be predicted using this modelling [5, 6, 67]. These modes are of special importance in the bow shock region of the

earth and the sheath region, since different modes can trigger different response in these regions from the interaction with the solar wind as shown in fig. 6.1. The knowledge gained during the course of this thesis will help in studying surface waves on a larger and more scale in future.
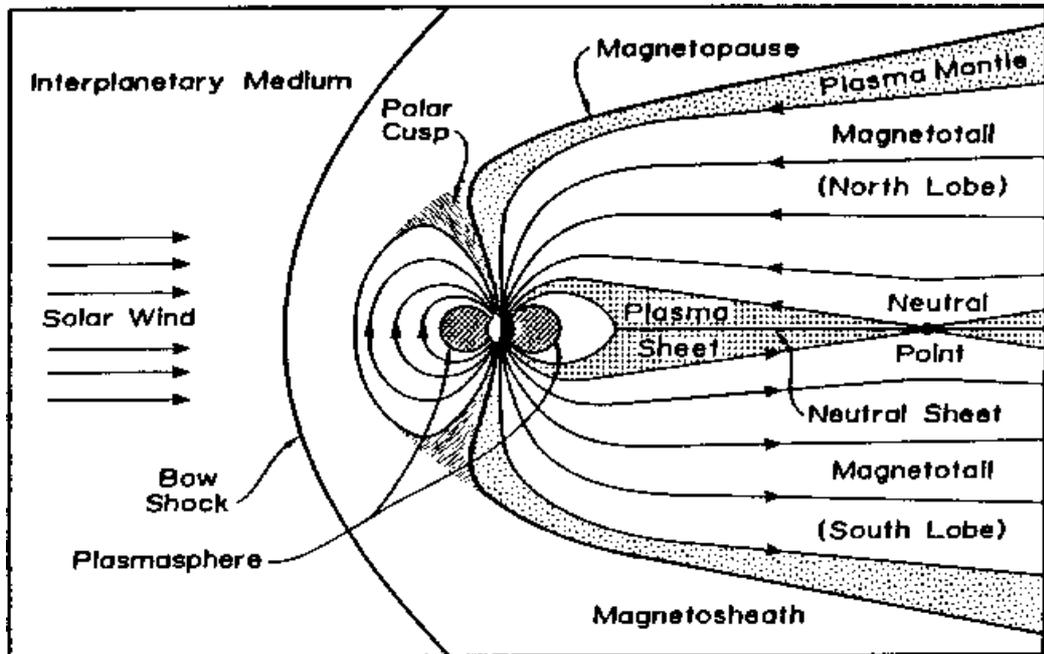

Fig. 6.1: Solar wind bow shock interaction and plasma sheath region.

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